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Deflection Solutions for Edge Stiffened Plates

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Abstract

The deflection of thin rectangular plates loaded by point loads and stiffened by elastic beams is determined using an infinite series approach. Both bending and torsional beam stiffness is included in the formulations. Solutions are obtained and results presented for a variety of plate aspect ratios and beam stiffness values. Comparisons of the results obtained with available solutions demonstrate good agreement, and parametric studies show the utility of this approach for various ratios of beam stiffness (bending and torisonal) to plate stiffness.

Introduction

Thin plates with beams affixed to the edges with the purpose of stiffening the assembly are commonly used in automotive, aerospace, marine, civil, and other practical applications. Theoretical expressions for the deflection of such plates with edge stiffening can be found in many classical textbooks such as Timoshenko [1] on plate theory. In general, solutions for thin elastic plates are given in terms of infinite series solutions of the governing differential equation. The solutions for edge stiffened plates are obtained by combining the differential equation governing the plate deflection with appropriate differential equation governing the beam deflection and enforcing compatibility conditions. In this way, both the bending and torsional stiffness of the beams can be included.

Many researchers have investigated stiffened plates, using a variety of approaches. A recent review by Satsangi and Mukhopaadhyay [2] of developments in static analysis of stiffened plates listed 104 papers in this area. They grouped the idealizations into three general classes as orthotropic plate theory, grillage theory, and plate beam idealization, and reviewed solution procedures. Concentrically and eccentrically stiffened plates have also been studied using a finite element approaches [3,4,5], and by boundary element techniques [6,7,8].

Although the equations are well known, the actual analytical solution of practical problems remains a challenge, due to the complexity of the equations involved. As such, to the knowledge of the authors the details of a complete general solution have not been published. Therefore, in the current paper we present the general solution for the displacement of thin elastic plates with arbitrary values for both flexural and torsional edge stiffeness under various loading conditions. In addition, numerical solutions are given for edge stiffness values ranging from zero (free) to infinity (completely rigid). The symbolic math toolbox in the computer application Matlab is used to facilitate the calculations.

The analytical solutions presented in this paper allow calculation of deflection of arbitrarily loaded and supported rectangular plates for a wide variety of design and analysis situations. Further, since the limited number of currently available solutions for edge stiffened plates have been used in the development of finite element and boundary element methods, the solutions presented herein may also be useful in the further development of these formulations.

Rectangular Plate Equations

The governing differential equation for the middle surface deflection of a thin, homogeneous, isotropic plate is[1]:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad \text{equation (1)}$$

where:

w=w(x,y)	is the plate deflection as a function of x and y
q=q(x,y)	is the applied distributed load, also as a function of x and y
$D = \frac{Eh^3}{12(1-v^2)}$	is the plate stiffness, with
E	the modulus of elasticity of the plate material
v	Poisson's' ratio for the plate material
h	the plate thickness

This equation is valid for flat plate structures with thickness that is small in relation to the width and length dimensions. In addition, the applied loads and resulting deflection of the plate are both assumed to be perpendicular to the plane of the plate. There are several other assumptions underlying equation (1), detailed the reference by Timoshenko and Woinowsky-Krieger[1]:

1. Deflections are small compared to the thickness of the plate.

2. The edges of the plate are free to move in the plane of the plate.
3. The midplane of the plate is a plane of zero strain (neutral axis).
4. The loads are such that the stresses are below the yield strength of the plate material, so that the material behaves elastically.

Edge Stiffened Plates

In this paper deflection solutions for rectangular plates with concentric edge stiffening beams on two opposite sides, loaded by a concentrated point load are found. Figure 1 shows the geometry under consideration.

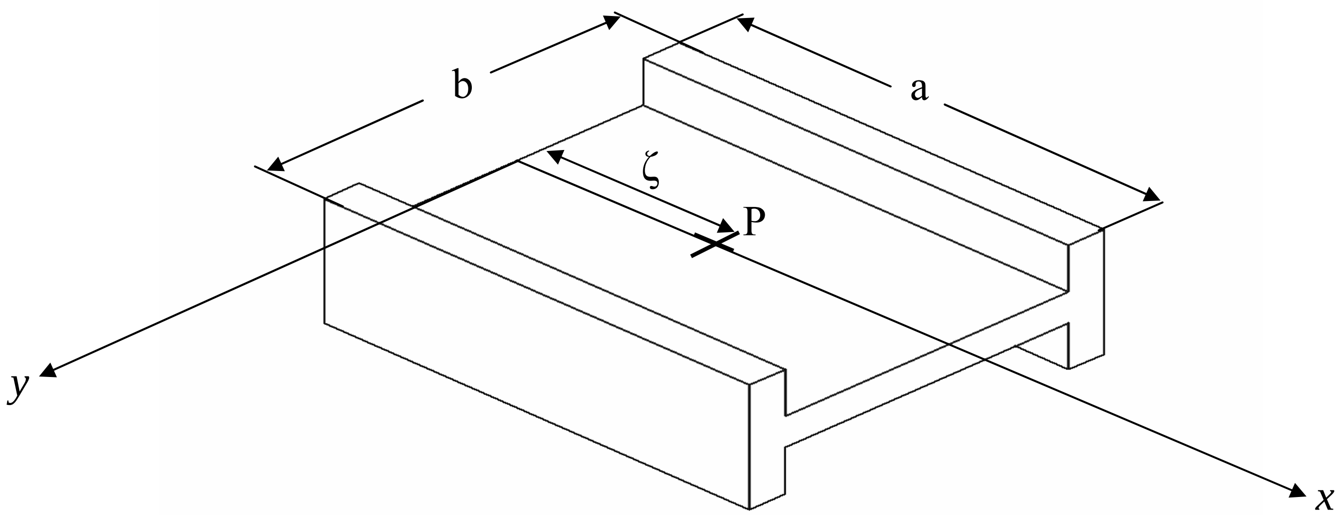

Figure 1.
Edge Stiffened Plate

The boundary conditions on the sides of the plate given by $x = 0$ and $x = a$ are assumed to be simply supported. This condition requires that the vertical deflection and the bending moment at any point along these two edges equal zero.

The boundary conditions for the plate edges at $y = -b/2$ and $y = b/2$ with attached stiffeners are more complex. The deflection along this edge is obtained from the compatibility requirement between the stiffener and the plate. In other words, the deflection of the beam must equal the deflection of the plate, resulting in the following boundary condition:

$$(EI)_b\left(\frac{\partial^4 w}{\partial x^4}\right)_{y=b/2} = D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2}+(2-\nu)\frac{\partial^2 w}{\partial x^2}\right]_{y=b/2} \qquad \text{equation (2)}$$

with:

$(EI)_b$ equal to the bending stiffness of the beam, given as the product of E, the modulus of elasticity of the curb material, and I, the second moment of inertia of the beam cross section.

In equation 2, the term of the left-hand side represents the beam deflection (from elementary beam theory), and the right hand term is the deflection of the plate at the edges where $y = b/2$.

The second boundary condition is obtained by considering the twisting of the stiffened edge. Again, compatibility requires that the rotation of the plate equal the twisting of the beam. The resulting boundary condition is:

$$(GJ)_b \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=b/2} = D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right]_{y=b/2} \qquad \text{equation (3)}$$

with:

$(GJ)_b$ equal to the torsional stiffness of the beam, given as the product of G, the shear modulus of the beam material, and J, the polar moment of inertia of the beam cross section. Note that the shear modulus can be expressed in terms of the modulus of elasticity and Poisson's ratio as:

$$G_b = \frac{E_b}{2(1+\nu)}$$

In equation 3, the term of the left-hand side represents the beam rotation (from elementary beam theory), and the right hand term is the rotation of the plate at the edges where $y = b/2$.

The boundary conditions on the simply supported edge are represented by:

$$w\big|_{x=0,a} = 0 \qquad \text{equation (4)}$$

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=0,a} = 0 \qquad \text{equation (5)}$$

General Solution

We seek a general solution of equation (1) subject to the boundary conditions given in equations (2-5). This solution can most easily be accomplished by constructing the total deflection *w* as the sum of two parts:

$$w = w_1 + w_2 \qquad \text{equation (6)}$$

Where w_1 is the particular solution, which depends upon the location and distribution of the loads applied to the plate, while w_2 is the homogeneous solution, selected to satisfy the boundary conditions.

Symmetric Case

To illustrate this approach, we consider a rectangular plate with length *a*, width *b*, and thickness *h* as shown in Figure 2. The two edges (at $x = 0$ and $x = a$) are simply supported and the other two edges (at $y = 0$ and $y = b$) are supported by elastic beams. For the case of a concentrated load *P* located by the coordinate ξ on the axis of symmetry *x* as shown in Figure 2, the deflection can be expressed by a single infinite series solution[1] as:

$$w_1 = \frac{Pa^2}{2\pi^3 D}\sum_{m=1}^{\infty}\left[Y_1\right]\sin\frac{m\pi x}{a} \qquad \text{equation (7)}$$

where:

$$Y_1 = \left[\left(1+\alpha_m \tanh\alpha_m\right)\sinh\frac{\alpha_m}{b}\left(b-2y\right)-\frac{\alpha_m}{b}\left(b-2y\right)\cosh\frac{\alpha_m}{b}\left(b-2y\right)\right]\frac{\sin\frac{m\pi\xi}{a}}{m^3\cosh\alpha_m}$$

with

$$\alpha_m = \frac{m\pi b}{2a}$$

This displacement solution is valid for $y \geq 0$, and will be symmetric about the x-axis. This expression satisfies the governing differential equation (1), as well as the simply supported boundary conditions at $x = 0$, and $x = a$ given by equations 4 and 5.

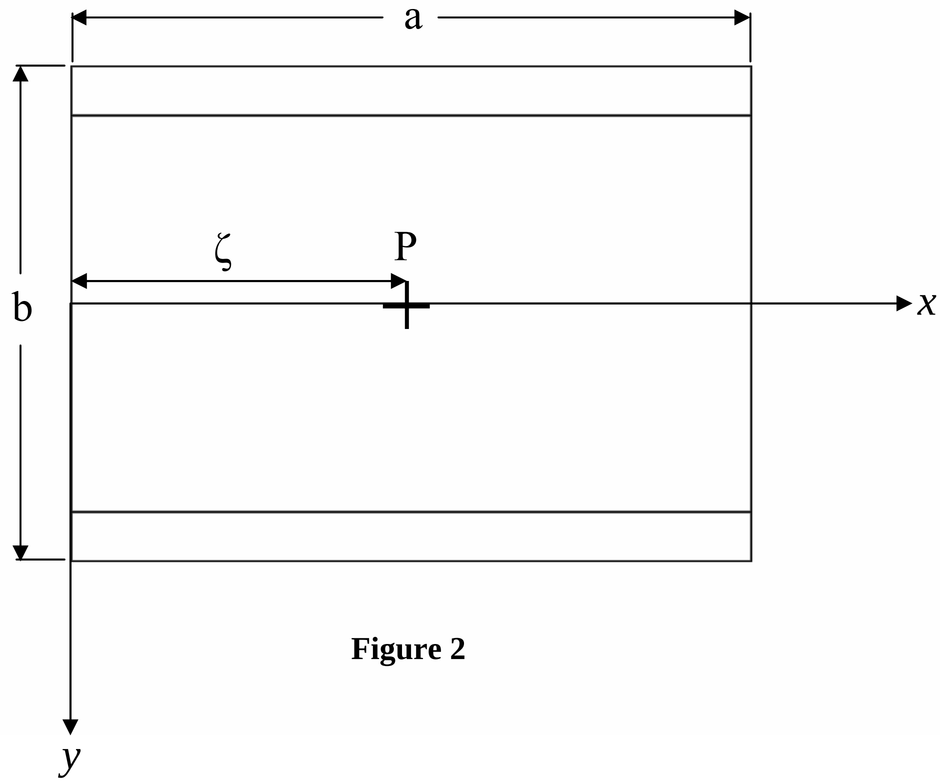

Figure 2

The solution w_2 must satisfy the homogeneous equation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0$$

we choose a solution of the form:

$$w_2 = \frac{Pa^2}{2\pi^3 D}\sum_{m=1}^{\infty}[Y_2]\sin\frac{m\pi x}{a} \qquad \text{equation (8)}$$

with:

$$Y_2 = A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$

The function w_2 evidently satisfies the boundary conditions at $x = 0$ and $x = a$. We must find suitable coefficients A_m and B_m such that the boundary conditions on edges at

y = *-b/2* and y = *b/2* given by equations (2) and (3) are satisfied by the total solution, $w = w_1 + w_2$, given by:

$$w = \frac{Pa^2}{2\pi^3 D}\sum_{m=1}^{\infty}\left[Y_1 + Y_2\right]\sin\frac{m\pi x}{a} \qquad \text{equation (9)}$$

The solution procedure is as follows:

1. Calculate appropriate derivatives of the assumed deflection solution (equation 9) and substitute these derivatives into boundary condition equations (2) and (3).
2. Solve equations (2) and (3) simultaneously for the constants A_m and B_m.
3. Insert A_m and B_m back into equation (9) to obtain the infinite series expression for plate deflection.
4. Sum an appropriate number of terms of the series to calculate the desired plate displacement.

The above steps represent a significant analytical challenge. Therefore, the Matlab[9] software package, which has the capability to perform symbolic computations, was used to assist with the rather tedious calculations in step one and two. Complete solutions for both A_m and B_m are provided in Appendix A.

Although equation (9) represents an infinite series solution, in practice it converges very rapidly so that only the first five or six terms are required for convergence. Again, software was written using the Matlab programming language to perform the calculations. In this way any variety of plate dimensions, beam properties, and loads can be investigated.

Examples

By appropriate selection of beam stiffness parameters, the above solution can be used for a wide variety of practical situations. The following two cases, for which established solutions exist, are considered:

1. Plate simply supported on all four sides.
2. Plate simply supported on two parallel sides and fixed on the other two sides.

By comparing existing solutions for these two cases to those obtained from equation 9 above, the solutions obtained in this work can be verified.

Case 1: Plate Simply Supported on all Four Sides

The deflection given by equation 9 is required by virtue of the selected functions to have zero deflection on the sides given by $x = 0$ and $x = a$, representing simple supports. By setting the ratio of beam to plate stiffness as an arbitrarily large number, combined with a beam torsional stiffness of zero, the plate effectively becomes simply supported on the beam sides as well. Accordingly, we set the non-dimensional bending and torsional stiffness ratios to the following values:

$$\frac{E_b I_b}{Da} = 10^6$$

$$\frac{G_b J_b}{Da} = 0$$

Software was written to use equation 9 to calculate the plate deflection over a number of points across the plate. Figure 3 shows the displaced shape obtained a square plate subject to a center load.

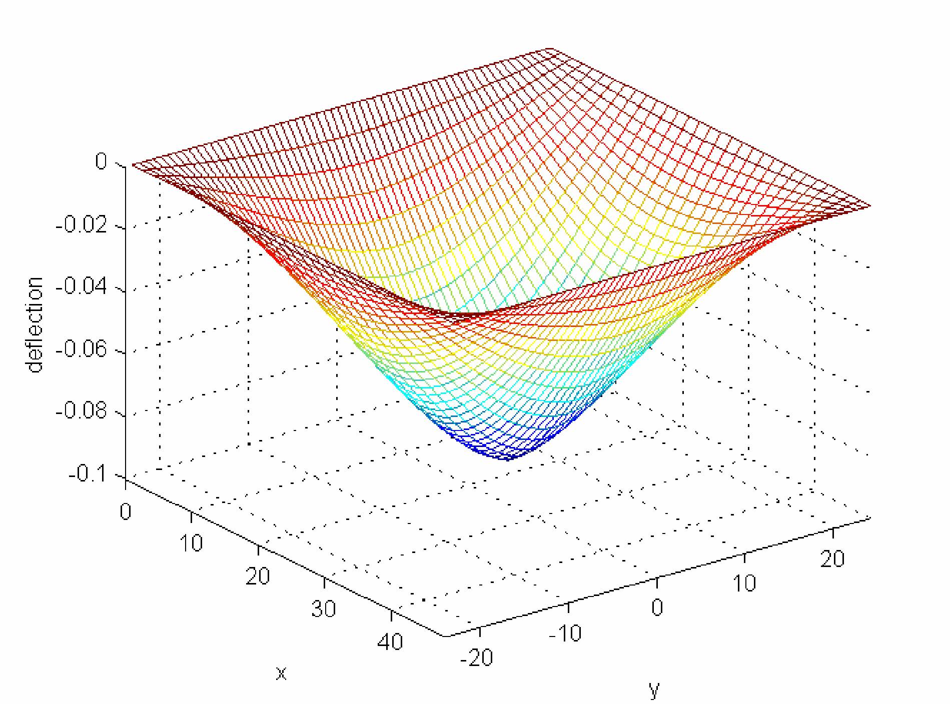

Figure 3. Square plate, center load, GJ/Da=0, EI/Da=∞

The solution for a simply supported rectangular plate subject to a central point load is well known. The maximum deflection at the center of the plate is given in [1] as:

$$w_{\max} = \alpha \frac{Pa^2}{D} \qquad \text{equation (10)}$$

For a square plate, the *b/a* ratio is 1 and the maximum deflection from equation 10 is identical to that obtained through the series solution of equation 9. To further validate the solution obtained in this work, the coefficient α is calculated from equation 9 above and compared for to the accepted solution from [1] for various values of the ratio *b/a* in Table 1. In general there is excellent agreement between the two solutions, with maximum differences less than 0.2 percent.

Table 1
Maximum non-Dimensional Deflection for Plate Simply Supported on all Four Sides.

b/a	1.0	1.1	1.2	1.4	1.6	1.8	2.0	3.0
Ref [1]	0.01160	0.01265	0.01353	0.01484	0.01570	0.01620	0.01651	0.01690
Eqn 9	0.01160	0.01267	0.01356	0.01487	0.01570	0.01621	0.01652	0.01693

Case 2: Plate simply supported on two parallel sides, fixed on the other two sides
By setting both the bending and torsional stiffness of the beam to high values, the beams in effect become fixed supports. Thus, setting

$$\frac{E_b I_b}{Da} = 10^6$$

$$\frac{G_b J_b}{Da} = 10^6$$

The maximum deflection of a rectangular plate fixed at $y=+/-b/2$ and simply supported on the other two sides and loaded in the center is given in [1] as:

$$w_{\max} = \alpha \frac{Pb^2}{2\pi^3 D} \qquad \text{equation (11)}$$

Figure 4 shows the deflected shape obtained from equation 9.

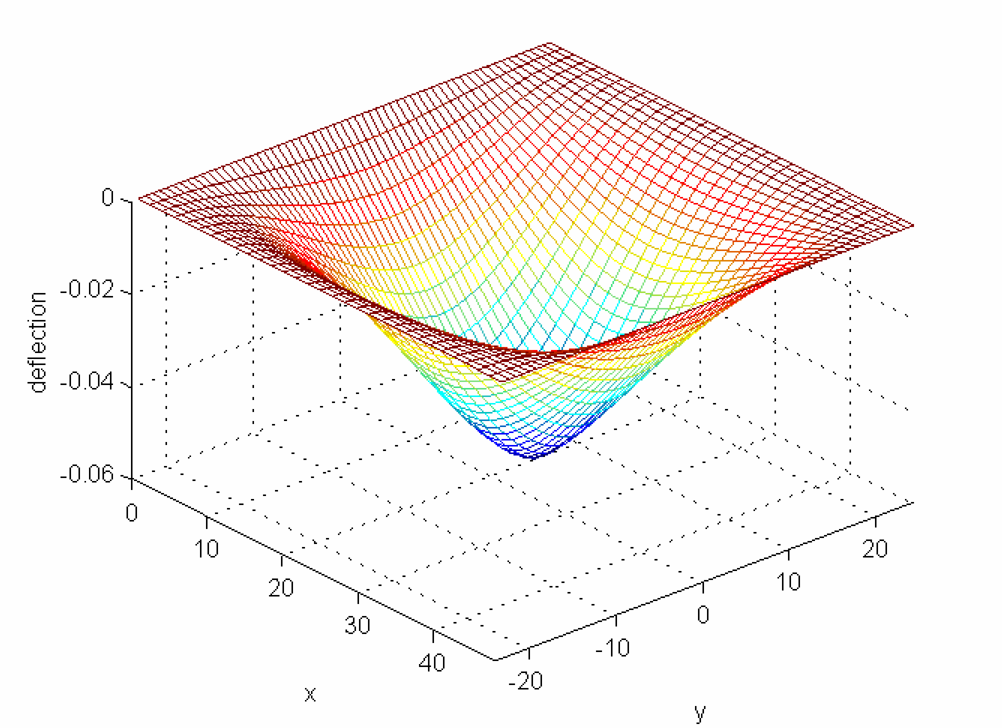

Figure 4. Square plate, center load, $GJ/Da=\infty$, $EI/Da=\infty$

Rearranging equation 11 so that α represents the non-dimensional displacement for the present case:

$$\alpha = \frac{w_{\max} 2\pi^3 D}{Pb^2}$$

Table 2 shows the results for the calculation of the coefficient α from equation 9 for various values of the ratio *b/a*. Once again, excellent agreement with the accepted solution is seen.

Table 2

b/a	2.0	1.0	0.5	0.33
Ref [1]	0.238	0.436	0.448	0.449
Eqn 9	0.238	0.437	0.450	0.449

Maximum non-Dimensional Deflection for Plate simply supported on two parallel sides, fixed on the other two sides.

Two other example cases are also illustrated. Figures 5 shows the displaced shape for a square plate with both bending and torsional beam stiffness set to zero, simulating free edges. Figure 6 shows the displaced shape with high torsional stiffness and zero bending stiffness, simulating a guided support (no rotation).

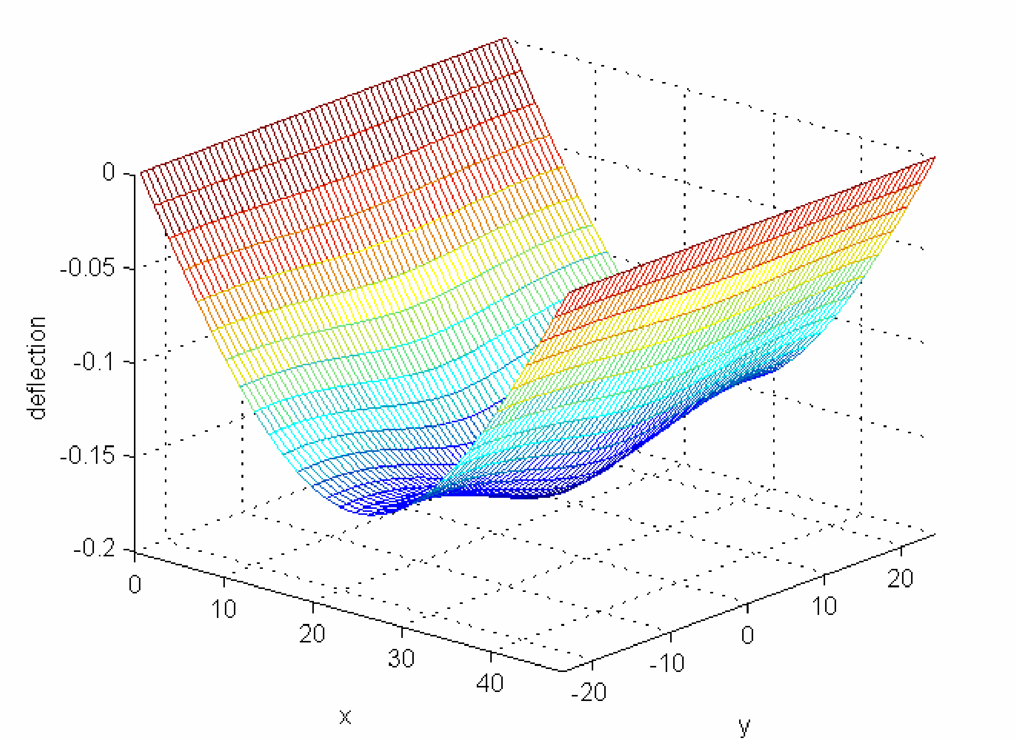

Figure 5. Square plate, center load, G*J/Da=0*, *EI/Da=0*

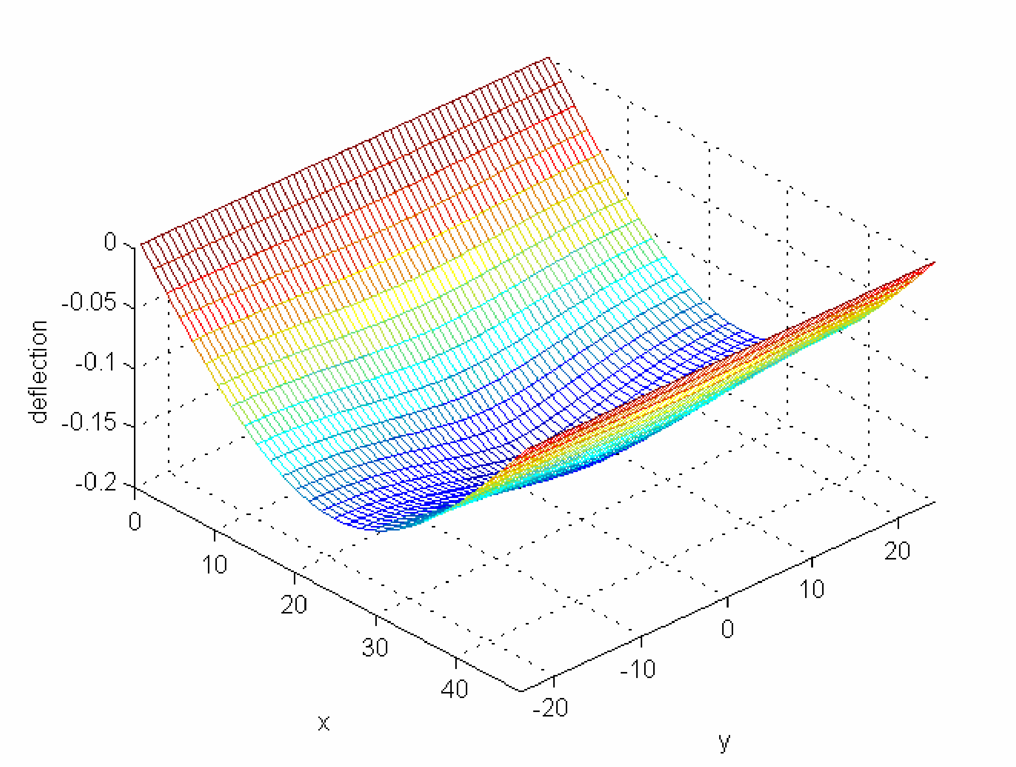

Figure 6. Square plate, center load, G*J/Da*= ∞, *EI/Da=0*

Parametric Studies

The dependence of the deflection on the bending and torsional stiffness of the beams is demonstrated through a parametric study in which the ratios of the beam (bending and torsional) stiffness to the plate stiffness is varied. A square plate, 60 inches on a side, 0.320 inches thick, with a modulus of elasticity of 10e6 psi and poisson's ratio of 0.33, and loaded with a single load in the center, was used for this study. The twenty two cases studied are listed in Table 3, along with the maximum deflection obtained. Figures 7 through 14 illustrate the dependence of deflection on beam stiffness for this parametric study.

Table 3
Results for Parametric Study of Stiffened Plate.

Case	Bending Ratio (br) *EI/Da*	Torsion Ratio (tr) *GJ/Da*	w_{max}
1	*0*	*0*	*.1748*
2	*0.1*	*0*	*.1533*
3	*1.0*	*0*	*.1078*
4	*10.*	*0*	*.0898*
5	*10000.*	*0*	*.0872*
6	*0*	∞	*.1744*
7	*0.1*	∞	*.1527*
8	*1.0*	∞	*.0911*
9	*10.*	∞	*.0583*
10	*100.*	∞	*.0535*
11	*1000.*	∞	*.0530*
12	*10000.*	∞	*.0529*
13	*0*	*0*	*.1748*
14	*0*	*0.1*	*.1747*
15	*0*	*1.0*	*.1745*
16	*0*	*10.*	*.1744*
17	∞	*0*	*.0872*
18	∞	*0.1*	*.0819*
19	∞	*1.0*	*.0650*
20	∞	*10.*	*.0547*
21	∞	*100.*	*.0531*
22	∞	*10000.*	*.0529*

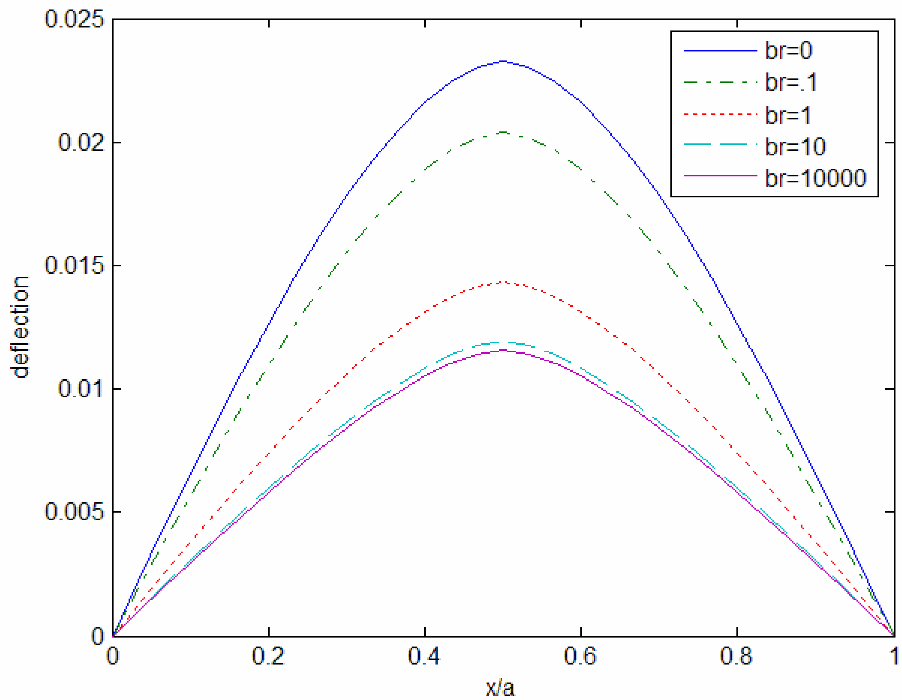

Figure 7. Centerline deflection ($y=0$), square plate, center load, G$J/Da=0$, various EI/Da ratios.

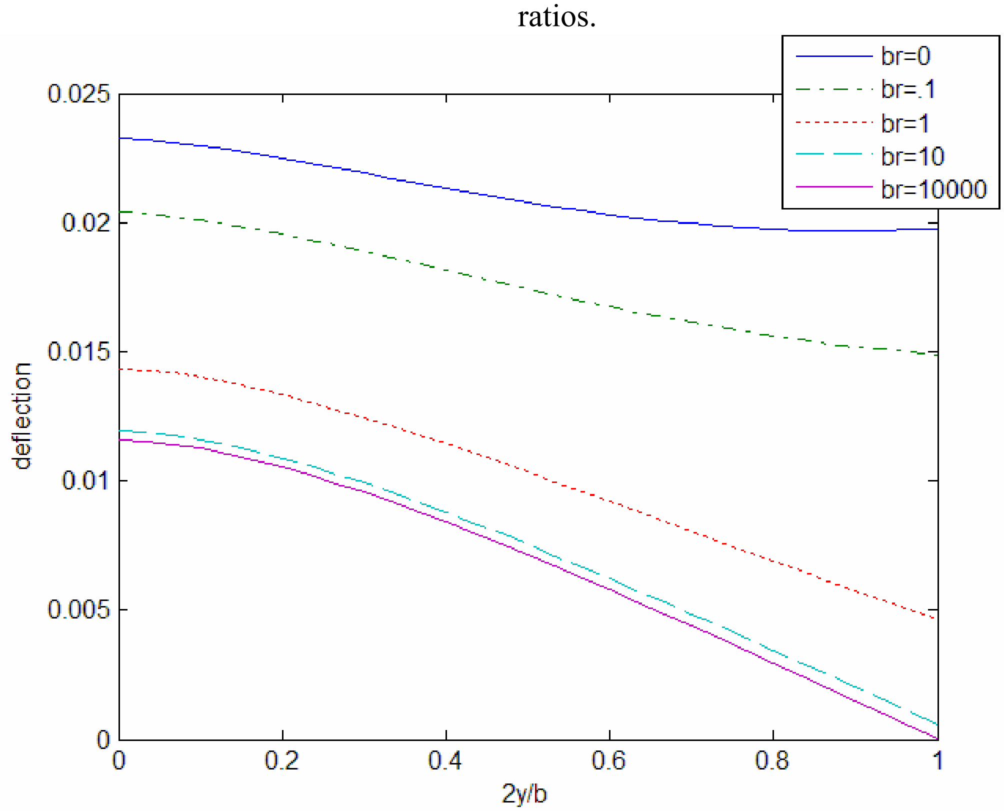

Figure 8. Centerline deflection ($x=a/2$), square plate, center load, $GJ/Da=0$, various EI/Da ratios.

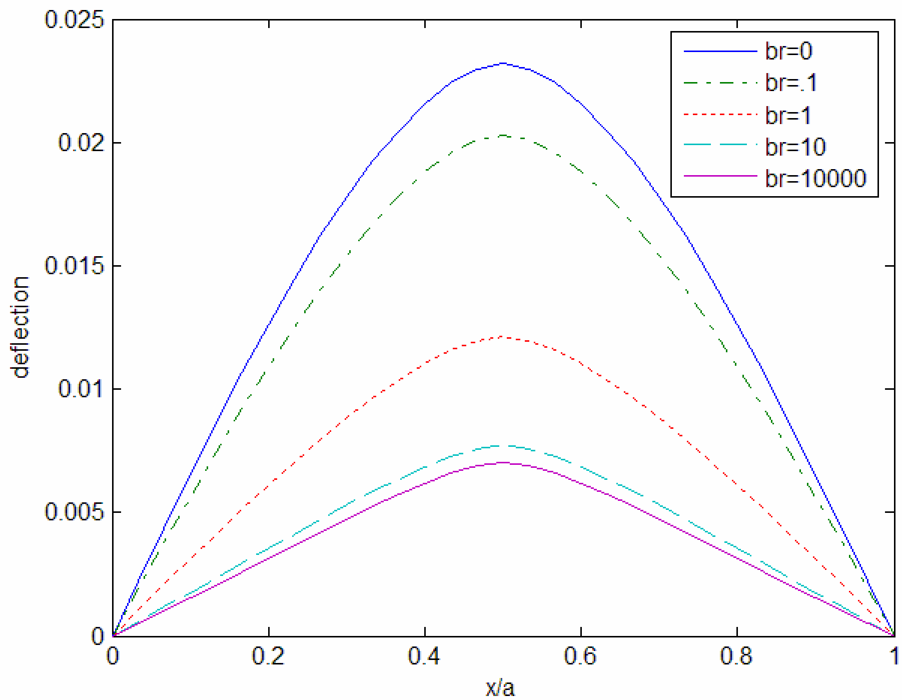

Figure 9. Centerline deflection ($y=0$), square plate, center load, G$J/Da=\infty$, various EI/Da ratios.

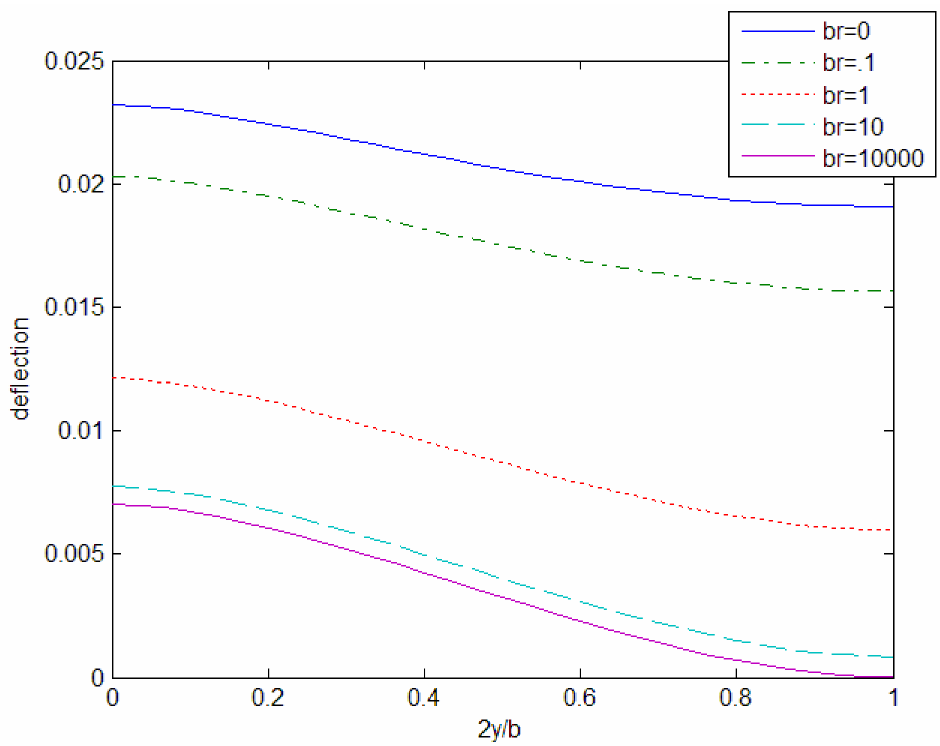

Figure 10. Centerline deflection ($x=a/2$), square plate, center load, G$J/Da=\infty$, various EI/Da ratios.

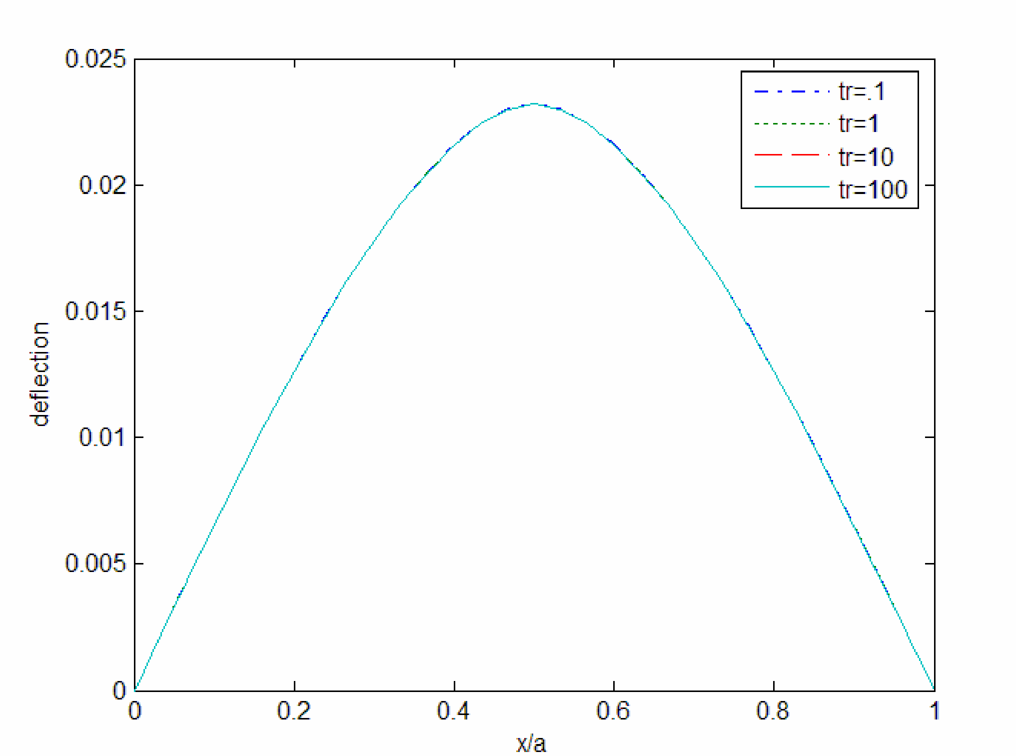

Figure 11. Centerline deflection (*y=0*), square plate, center load, *EI/Da=0*, various *GJ/Da* ratios.

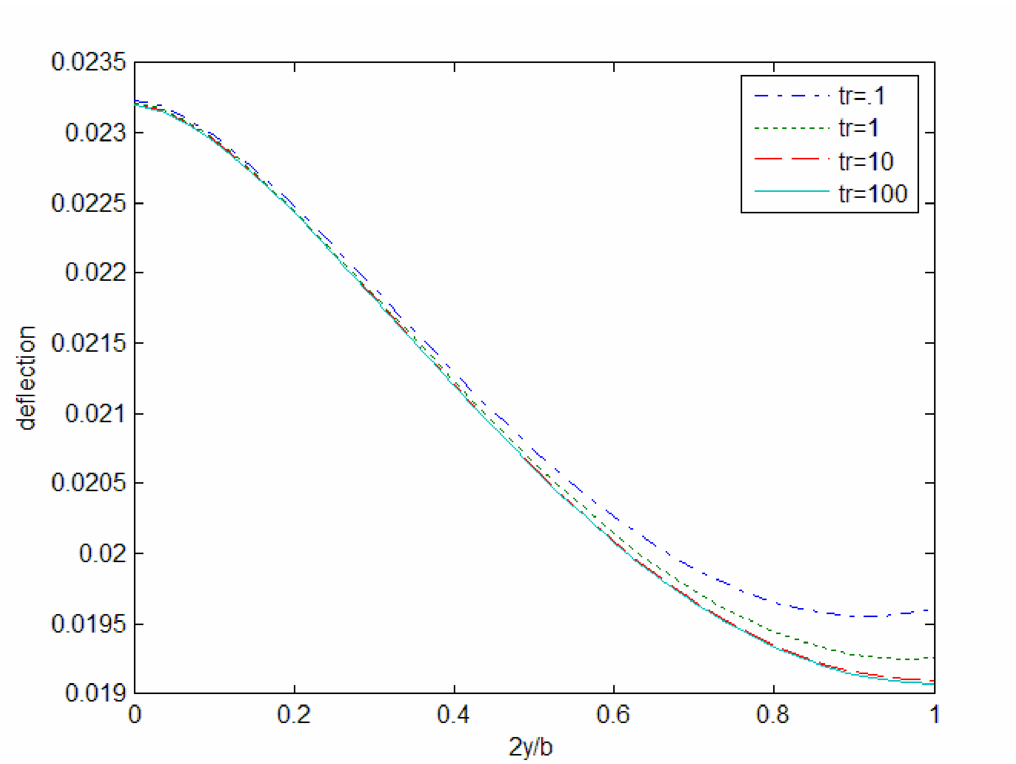

Figure 12. Centerline deflection (*x=a/2*), square plate, center load, *EI/Da=0*, various *GJ/Da* ratios.

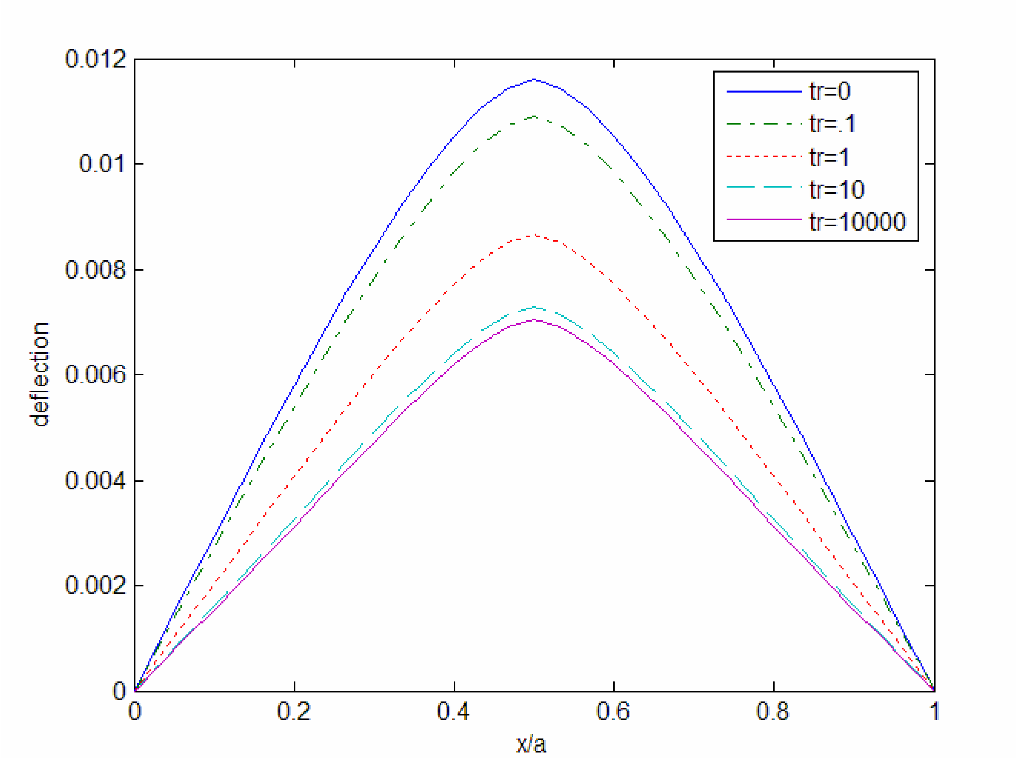

Figure 13. Centerline deflection (*y=0*), square plate, center load, *EI/Da*= ∞, various *GJ/Da* ratios.

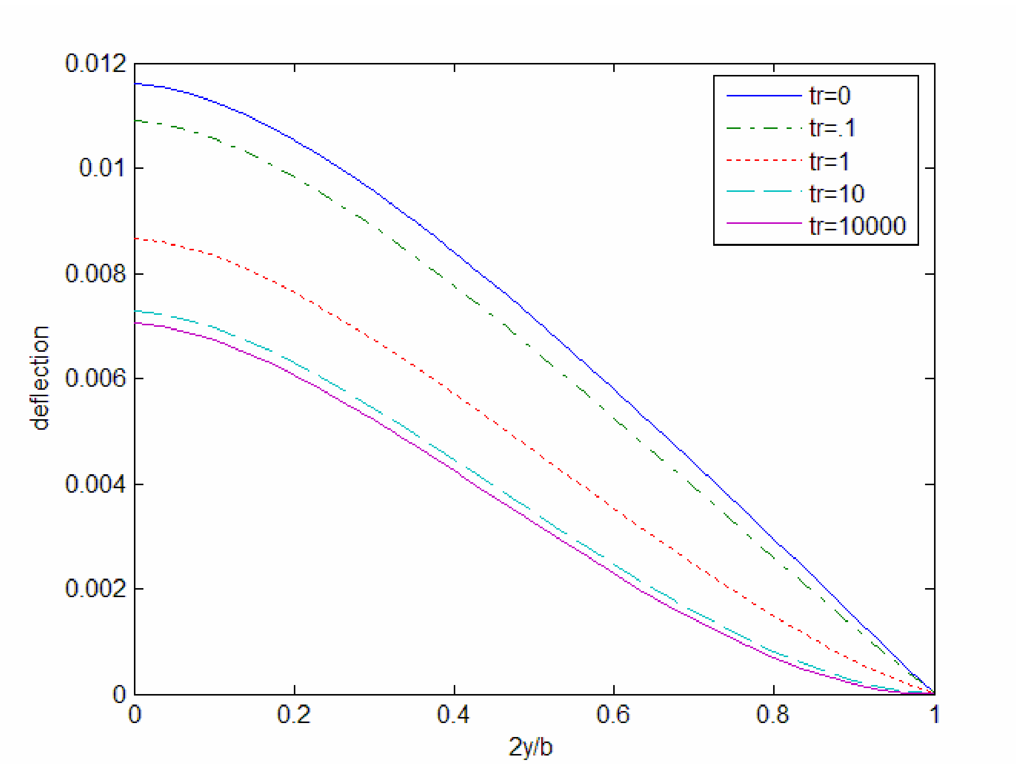

Figure 14. Centerline deflection (*x=a/2*), square plate, center load, *EI/Da* = ∞, various *GJ/Da* ratios.

Conclusions

Deflection solutions for edge stiffened plates using an infinite series approach have been obtained. Although previous authors have outlined a solution procedure, in this work complete expressions for all terms have been found. Comparison with available analytical solutions show good agreement, and parametric studies reveal the utility of the method for arbitrary beam bending and torsional stiffness values.

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Biographies

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Appendix A

Expressions for A_m and B_m

```
alm=(m*pi*b)/(2*a);
    c1=1+alm*tanh(alm);
    c2=alm/b;
    c3=(sin((m*pi*z)/a)/(m^3*cosh(alm)));
    c6=(m*pi)/a;


    am=2*(-4*c6^2*D*c1*c2^2*G*J*exp(c6*b)*b+12*c6*exp(c6*b)*b*D^2*c2^2-
4*c6*exp(c6*b)*b*D^2*c1*c2^2 ...
    +4*c6*b*D^2*c1*c2^2+4*c6*exp(c6*b)*b*D^2*c9*c1*c2^2-
12*c6*exp(c6*b)*b*D^2*c9*c2^2-4*c6*b*D^2*c9*c1*c2^2 ...
    +EB*I*c6^5*b*exp(c6*b)*G*J-EB*I*c6^5*b*exp(c6*b)*G*J*c1-
6*G*J*exp(c6*b)*c6^3*D+6*G*J*c1*exp(c6*b)*c6^3*D ...
    -16*D^2*exp(c6*b)*c1*c2^2+4*D^2*exp(c6*b)*c8*c6^2*c1+6*G*J*c6^3*D-
6*G*J*c1*c6^3*D-16*D^2*c1*c2^2+48*D^2*c2^2 ...
    +4*D^2*c8*c6^2*c1+8*G*J*c6*D*c1*c2^2-24*G*J*c6*D*c2^2-
4*D^2*c8*c6^2-8*G*J*c6*exp(c6*b)*D*c1*c2^2+24*G*J*c6*exp(c6*b)*D*c2^2
...
    -4*D^2*exp(c6*b)*c8*c6^2+c6^4*D*b*exp(c6*b)*G*J*c1-
c6^4*D*b*exp(c6*b)*G*J-c6^4*D*b*G*J+c6^4*D*b*G*J*c1-c6^3*b*D^2*c8*c1
...
    -
c6^3*b*D^2*c9*c8+c6^3*b*D^2*c8+c6^3*b*D^2*c9*c8*c1+12*c6^2*b*G*J*D*c2^2
-4*c6^2*b*G*J*D*c1*c2^2-12*c6*b*D^2*c2^2 ...
    +12*c6*b*D^2*c9*c2^2 ...
    -EB*I*c6^5*b*G*J+EB*I*c6^5*b*G*J*c1-c6^3*exp(c6*b)*b*D^2*c9*c8*c1-
c6^3*exp(c6*b)*b*D^2*c8+c6^3*exp(c6*b)*b*D^2*c8*c1 ...

+c6^3*exp(c6*b)*b*D^2*c9*c8+48*D^2*exp(c6*b)*c2^2+12*c6^2*D*c2^2*G*J*ex
p(c6*b)*b)*c3*c2*exp(1/2*c6*b)/c6^3/(2*D*c6*exp(2*c6*b)*EB*I ...
    +2*D*c6*EB*I-
G*J*c6^2*EB*I+G*J*c6^2*exp(2*c6*b)*EB*I+D^2*exp(2*c6*b)-
2*D^2*exp(c6*b)*c8*c6*b+D^2*exp(2*c6*b)*c8 ...
    +2*D^2*exp(c6*b)*c6*b+2*G*J*c6*exp(2*c6*b)*D-3*D^2*c9*exp(2*c6*b)-
D^2*c8-D^2+2*G*J*c6^3*exp(c6*b)*EB*I*b+2*G*J*c6*D+3*D^2*c9 ...
    -
4*G*J*c6*exp(c6*b)*D+2*D^2*c9*exp(c6*b)*c8*c6*b+D^2*c9*exp(2*c6*b)*c8-
2*D^2*c9*exp(c6*b)*c6*b-D^2*c9*c8+4*D*c6*exp(c6*b)*EB*I);


    bm=-4*c3*c2*exp(1/2*c6*b)*(-
G*J*c1*exp(c6*b)*c6^4*EB*I+G*J*exp(c6*b)*c6^4*EB*I-
G*J*exp(c6*b)*c6^3*D+D^2*c9*exp(c6*b)*c8*c6^2 ...
    -12*D^2*c9*exp(c6*b)*c2^2-
D^2*c9*exp(c6*b)*c8*c6^2*c1+G*J*c1*exp(c6*b)*c6^3*D-
4*D^2*exp(c6*b)*c1*c2^2+4*D^2*c9*exp(c6*b)*c1*c2^2 ...
    +D^2*exp(c6*b)*c8*c6^2*c1+G*J*c6^3*D+G*J*c6^4*EB*I-G*J*c1*c6^3*D-
G*J*c1*c6^4*EB*I-4*D^2*c1*c2^2+12*D^2*c2^2+D^2*c8*c6^2*c1 ...
    +4*G*J*c6*D*c1*c2^2-12*G*J*c6*D*c2^2-D^2*c8*c6^2-
12*D^2*c9*c2^2+D^2*c9*c8*c6^2+4*D^2*c9*c1*c2^2-D^2*c9*c8*c6^2*c1 ...
```

```
    -4*G*J*c6*exp(c6*b)*D*c1*c2^2+12*G*J*c6*exp(c6*b)*D*c2^2-
D^2*exp(c6*b)*c8*c6^2+12*D^2*exp(c6*b)*c2^2)/c6^3/(2*D*c6*exp(2*c6*b)*E
B*I ...
    +2*D*c6*EB*I-
G*J*c6^2*EB*I+G*J*c6^2*exp(2*c6*b)*EB*I+D^2*exp(2*c6*b)-
2*D^2*exp(c6*b)*c8*c6*b+D^2*exp(2*c6*b)*c8+2*D^2*exp(c6*b)*c6*b ...
    +2*G*J*c6*exp(2*c6*b)*D-3*D^2*c9*exp(2*c6*b)-D^2*c8-
D^2+2*G*J*c6^3*exp(c6*b)*EB*I*b+2*G*J*c6*D+3*D^2*c9-
4*G*J*c6*exp(c6*b)*D ...
    +2*D^2*c9*exp(c6*b)*c8*c6*b+D^2*c9*exp(2*c6*b)*c8-
2*D^2*c9*exp(c6*b)*c6*b-D^2*c9*c8+4*D*c6*exp(c6*b)*EB*I);
```