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Stochastic FE Heat Transfer Analysis Applied to a Storage Container Design

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Abstract

Heat transfer from a water tank was computationally simulated by a finite element method and probabilistically evaluated in view of the several uncertainties in the performance parameters. Cumulative distribution functions and sensitivity factors were computed for overall heat transfer rate due to the thermodynamic random variables. These results can be used to quickly identify the most critical design variables in order to optimize the design and make it cost effective. The analysis leads to the selection of the appropriate measurements to be used in heat transfer and to the identification of both the most critical measurements and parameters.

Introduction

Engineers always face uncertainties in design, whether it is in the prediction of future loads, variability of material properties or uncertainties in predicting system response under load. For aging structures, probabilistic mechanics provides the means to quantify the safety of the structure. For new design, probabilistic mechanics provides the means to explicitly treat uncertainties to achieve truly optimal design. Probabilistic methods provide the engineer with a way to quantify uncertainties and treat all problem uncertainties consistently. The traditional approach of using arbitrary design safety factors does not provide a means to quantify the design reliability and can sometimes lead to unbalanced designs wherein some components are over-designed and some may be actually be under-designed.

Different kinds of analyses accounting for uncertainties can be carried out. Second moment analysis aims at characterizing the second-order statistical moments, i.e. means and variances of response quantities (displacements, strain and stress components, etc.) from those of the input variables. The perturbation method [1-2] and the weighted integral method [3-5] are in this category. On the other hand, first order reliability method (FORM) and second order reliability method (SORM) approximations and various simulation methods are commonly used in reliability analysis [6]. Because of the typically high level of reliability of civil structures, the failure probability is usually small (in the order of $10^{-2} – 10^{-6}$).

A probabilistic design system was developed by Fox [7] at Pratt and Whitney for the purpose of integrating deterministic design methods with probabilistic design techniques. Here, two different approaches were used for estimating uncertainty. A Monte Carlo approach was used on design codes that were judged to run relatively quickly. For more computationally intensive design codes, a second order response surface model in conjunction with Box-Behnken design experiments was used and then a Monte Carlo simulation was executed. Several researchers at NASA Glenn Research Center have applied the probabilistic design approaches to turbine engines and related systems. Chamis [8] developed a Probabilistic Structural Analysis Method (PSAM) using different distributions such as the Weibull, normal, log-normal etc. to describe the uncertainties in the structural and load parameters or primitive variables. Nagpal, Rubinstein and Chamis [9] presented a probabilistic study of turbopump blades of the Space Shuttle Main Engine (SSME). They found that random variations or uncertainties in geometry have statistically significant influence on the response variable and random variations in material properties have statistically insignificant effects.

To cost effectively accomplish the design task, we need to formally quantify the effect of uncertainties (variables) in the design. Probabilistic design is one effective method to formally quantify the effect of uncertainties. In the present paper, a probabilistic analysis is presented for the influence of measurement accuracy and apriori fixed parameter variations on the random variables for heat transfer from a pressure vessel. Small perturbation approach is used for the finite element methods to compute the sensitivity of the response to small fluctuations of the random variables present. The result is a parametric representation of the response in terms of a set of random variables with known statistical properties, which can be used to estimate the characteristics of the selected response variables such as heat transfer rate or temperature at a given point.

Analysis

Let us consider the steady state heat transfer. Figure 1 shows the coordinate system and the model used. An energy balance yields

$$\frac{\partial}{\partial x}\left(Kt\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(Kt\frac{\partial T}{\partial y}\right) = 0 \,. \qquad (1)$$

In the above equation, T is the temperature, T_∞ the ambient temperature, K the thermal conductivity of the fin material, t the thickness of the fin, h the heat transfer coefficient and x and y the coordinate directions.

The boundary condition is given by:

$$T(0,y) = T_0. \qquad (2)$$

where T_0 is the inside surface temperature of the tank.
Along the sides where convection occurs to the outside fluid, we have

$$Kt\left(\frac{\partial T}{\partial x}\right)n_x + Kt\left(\frac{\partial T}{\partial y}\right)n_y + ht(T - T_\infty) = 0 \qquad (3)$$

Equation (1) together with the boundary conditions was solved by the finite element numerical method.

Finite Element Solution

Let us consider a two-dimensional partial differential equation of the form

$$\frac{\partial}{\partial x}\left[K_x(x,y)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y(x,y)\frac{\partial T}{\partial y}\right] + P(x,y)T + Q(x,y) = 0. \qquad (4)$$

The above equation is valid over an area A. We assume that on a portion of the boundary L_1, $T = T_0\,(x,y)$.

On the remainder of the boundary, labeled L_2, the general derivative boundary condition is specified in the form

$$K_x(x,y)\frac{\partial T}{\partial X}\,n_x + K_y(x,y)\frac{\partial T}{\partial y}\,n_y + \alpha(x,y)T + \beta(x,y) = 0. \qquad (5)$$

Here, n_x and n_y are direction cosines of the outward normal to L_2. The form of the functional may be written as

$$I(T) = \iint_A \left[\frac{1}{2}K_x\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{2}K_y\left(\frac{\partial T}{\partial y}\right)^2 - \frac{1}{2}PT^2 - QT\right]dA + \int_{L_2}\left(\frac{\alpha T^2}{2} + \beta T\right)dL. \qquad (6)$$

For a simplex two-dimensional element, we have extremized the above functional with respect to the unknown nodal temperatures. The resultant element matrices are then obtained from the following relation:

$$\begin{Bmatrix} \dfrac{\partial I}{\partial T_i} \\ \dfrac{\partial I}{\partial T_j} \\ \dfrac{\partial I}{\partial T_k} \end{Bmatrix}^{(e)} = [B]^{(e)}[T]^{(e)} - [C]^{(e)} \qquad (7)$$

The element matrix $[B]^{(e)}$ and the element column $[C]^{(e)}$ may be written as

$$[B]^{(e)} = \begin{bmatrix} B_{ii} B_{ij} B_{ik} \\ B_{ji} B_{jj} B_{jk} \\ B_{ki} B_{kj} B_{kk} \end{bmatrix}, \quad [C]^{(e)} = \begin{Bmatrix} C_i \\ C_j \\ C_k \end{Bmatrix}^{(e)}, \tag{8}$$

Where

$$B_{ii} = \frac{K}{4A}(b_i^{\,2} + c_i^{\,2}) - \frac{PA}{6} + \frac{(\alpha L_{ij})_{sideij}}{3} + \frac{(\alpha L_{ki})_{sideki}}{3},$$

$$B_{ij} = \frac{K}{4A}(b_i b_j + c_i c_j) - \frac{PA}{12} + \frac{(\alpha L_{ij})_{sideij}}{6},$$

$$B_{ik} = \frac{K}{4A}(b_i b_k + c_i c_k) - \frac{PA}{12} + \frac{(\alpha L_{ki})_{sideki}}{3},$$

$$B_{jj} = \frac{K}{4A}(b_j^{\,2} + c_j^{\,2}) - \frac{PA}{6} + \frac{(\alpha L_{jk})_{sidejk}}{3} + \frac{(\alpha L_{ij})_{sideij}}{3},$$

$$B_{jk} = \frac{K}{4A}(b_j b_k + c_j c_k) - \frac{PA}{12} + \frac{(\alpha L_{jk})_{sidejk}}{6},$$

$$B_{kk} = \frac{K}{4A}(b_k^{\,2} + c_k^{\,2}) - \frac{PA}{6} + \frac{(\alpha L_{ki})_{sideki}}{3} + \frac{(\alpha L_{jk})_{sidejk}}{3},$$

$$C_i = \frac{QA}{3} - \frac{\beta L_{ij}}{2} - \frac{\beta L_{ki}}{2}, \; C_j = \frac{QA}{3} - \frac{\beta L_{jk}}{2} - \frac{\beta L_{ij}}{2}, \; C_k = \frac{QA}{3} - \frac{\beta L_{ki}}{2} - \frac{\beta L_{jk}}{2},$$

The element matrices were then assembled into the global matrices and vectors. The prescribed boundary conditions were implemented at the appropriate nodal points. The algebraic equations in the global assembled form were solved by the Gauss elimination procedure. These details may be found in reference [10].

Perturbation of the Heat Transfer Problem

The finite element solution for the heat transfer problem may be reduced to the following equation in the unperturbed state:

$$[B][T] = [C] \tag{9}$$

The perturbed problem involving small fluctuations of the random variables may be written as

$$[\hat{B}]\ [\hat{T}] = [\hat{C}] \quad (10)$$

where

$$[\hat{B}] = [B] + d[B]$$
$$[\hat{T}] = [T] + d[T]$$
$$[\hat{C}] = [C] + d[C] \quad (11)$$

Therefore, we may write equation (9) as

$$[B]d[T] = [C] - [\hat{B}][\hat{T}] - d[B]\ d[T]$$
$$\cong dx_i\ \frac{\partial[C]}{\partial x_i} - dx_i\ \frac{\partial[B]}{\partial x_i}[T] \quad (12)$$

In the last step in equation (12), we ignored the second order term d [B]. d [T]. Here, x_i are the random variables. A simple form of the iterative algorithm is given by:

$$[B]d[\hat{T}]^{n+1} = [\hat{C}] - [\hat{B}][\hat{T}]^n \quad (13)$$

$$[\hat{T}]^{n+1} = [\hat{T}]^n + d[\hat{T}]^{n+1} \quad (14)$$

In order to start the iteration, we may use

$$[\hat{T}]^0 = [T]$$

The effect of temperature-dependent properties may be included in equation (13). From equation (13), we may write:

$$[B]d[\hat{T}]^n = [\hat{C}] - [\hat{B}][\hat{T}]^{n-1} \quad (15)$$

From equations (13) and (15) we may write

$$[B]d[\hat{T}]^{n+1} = [B]d[\hat{T}]^n - [\hat{B}]\ d[\hat{T}]^n \quad (16)$$

From equation (16), we may write

$$d[\hat{T}]^{n+1} = [A]\ d[\hat{T}]^n \quad (17)$$

where $[A] = [I] - [B]^{-1}\ [\hat{B}]$ is the amplification matrix. The iterative process will remain stable if the spectral radius of the amplification matrix [A] is less than unity. This will be true when the imposed perturbations on the original element matrix are small.

Probability Functions

Attention is now directed to the implementation of this probabilistic formulation in the design process. The necessary transition from the mathematical formulation above to a probabilistic model that yields the information relevant for multi-variate decision-making is described in this section. There are two alternatives for this task.

The first joint probability density function introduced here is an analytical probability model for criteria whose univariate distributions and their corresponding means and standard deviations are known. All necessary information for the model can be generated by the traditional probabilistic design process, using its output of univariate criterion distributions. A particular model for two criteria with normal distributions, represented by equation (17), has been introduced by Garvey and Tuab. Garvey further generated models for two criteria with combinations of normal and lognormal distributions, which are summarized in reference [11].

$$f_{XY}(x, y) = \frac{1}{2\pi\,\sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\{\frac{1}{2\rho^2 - 2}[(\frac{x-\mu_x}{\sigma_X})^2 - 2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) + (\frac{y-\mu_Y}{\sigma_Y})^2]\}$$

(17)

Note that the only information needed for the Joint Probability Model consists of the means μ_X and μ_Y, the standard deviations σ_X and σ_Y, and the correlation coefficient ρ for the criteria X and Y. The model variables, x and y, are defined over the interval of all possible criterion values. The advantage of this model is the limited information needed, which makes it very flexible for use and application. For example, if only expert knowledge and no simulation/modeling is available in the early stages of design, educated guesses for the means, standard deviations, and the correlation coefficient can be used to execute the joint probability model. It also lends itself to use in combination with increasingly important fast probability integration (FPI) techniques.

Implementation of Probabilistic Procedure Using FPI. FPI is a probabilistic analysis tool that implements a variety of methods for probabilistic analysis. The procedure follows the steps given below:

1. Identify the independent and uncorrelated design variables with uncertainties.
2. Quantify the uncertainties of these design variables with probability distributions based on expert opinion elicitation, historical data or benchmark testing.
3. It is required that there is a response function that defines the relationship between the response and the independent variables.

4. The FPI uses the responses generated to compute the cumulative distribution functions (CDF)/probability density functions (PDF) and the corresponding sensitivities of the response.

Several methods are available in the FPI to compute a probabilistic distribution. In addition to obtaining the CDF/PDF of the response, the FPI provides additional information regarding the sensitivity of the response with respect to the primitive variables. They provide valuable information in controlling the scatter of the response variable. The random primitive variable with the highest sensitivity factor will yield the biggest payoff in controlling the scatter in that particular response variable. Such information is very useful to the test/design engineer in designing or interpreting the measured data.

Results and Discussion

The history of the iterative algorithm is illustrated by means of an example involving hydrostatic pressure and self weight, conduction and convection in a water tank as shown in Figure 1. The inner surface of the tank was maintained at 400°C, while the outer surface at 0°C. The tank was filled with liquid water. The tank was composed of concrete, and standard gravity was assumed. The present results for hoop stress on the inside surface of the tank subjected to hydrostatic pressure are compared with those of Zienkiewicz [12]. Table 1 shows the random variables and their mean values used. All random variables were assumed to be independent. A scatter of ±10 % was specified for all the variables. This variation amounted to two standard deviations. Normal distribution was assumed for all random variable scatters.

Maximum thermal stress location was determined from a pre-analysis of the water tank. This location was used to evaluate the cumulative distribution functions (CDF) and the sensitivity factors for thermal stress response. Temperature distribution in the tank is shown in Figure 2. A typical thermal stress distribution in the water tank is shown in Figure 3. CDF for the thermal stress is shown in Figure 4. The sensitivity factors for the thermal stress versus the random variables are shown in Figure 5. We observe that the modulus of elasticity, Poisson's ratio, coefficient of thermal expansion of the tank material and inside surface temperature of the tank have a lot of influence on the thermal stresses.

Conventional engineering design methods are deterministic. The components of a machine are considered as ideal systems and parameter optimizations provide single point estimates of the system response. In reality, many engineering systems are stochastic where a probability assessment of the results is required. Probabilistic engineering design analysis assumes probability distributions of design parameters, instead of mean values only. This enables the designer to design for a specific reliability and hence maximize safety, quality and cost. The approaches for incorporating probabilistic effects in design include the use of factors of safety, the use of the worst case design and the use of probabilistic design. Utilizing the uncertainties in the estimations, deterministic engineering design uses factors of safety to assure that the nominal operational condition does not come too close to the point where the system will fail. The approximation of minimum properties and maximum loads known as the absolute worst case gives information about this critical point. This approach limits the optimization capability of a system and fails to provide important information about the system lifetime.

A robust design is one that has been created with a system of design tools that reduce product or process variability while guiding the performance toward an optimal setting. Robustness means achieving excellent performance under a wide range of operating conditions. All engineering systems function reasonably well under ideal conditions, but robust designs continue to function well when the conditions are non-ideal. Analytical robust design attempts to determine the values of design parameters, which maximize the reliability of the product without tightening the material or environmental tolerances. Probabilistic design and robust design go hand in hand. In order to determine the domains of stability, the system has to be analyzed probabilistically.

Conclusion

In this paper, a non-deterministic method has been developed to support reliability-based design. The novelty in the paper is the probabilistic evaluation of the finite element solution for heat transfer. Cumulative distribution functions and sensitivity factors were computed for heat loss due to the random variables. The inside tank temperature, modulus of elasticity, Poisson's ratio and the coefficient of thermal expansion of the tank material have a lot of influence on the thermal stresses. Evaluating the probability of risk and sensitivity factors will enable the identification of the most critical design variable in order to optimize the design and make it cost effective.

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Biography

Asad Salem is currently an associate professor of Mechanical Engineering and Engineering Technology Program Coordinator at Texas A&M University-Corpus Christi. His areas of interest are Computational Fluid Dynamics, Heat transfer and Energy Systems. He has over 25 years of experience as an engineer, researcher, and educator.

Yousri Elkassabgi is a professor of mechanical engineering at Texas A&M University-Kingsville (TAMUK). He specializes in heat transfer with or without phase change. He conducts research in areas of energy conservation; boiling heat transfer, and thermal hydraulics of nuclear reactors. He was the Director and founder of the Industrial Assessment Center IAC (formerly the Energy Analysis and Diagnostic Center, EADC) at TAMUK.

Table 1: Random Variables

Mass Density	4.5620000E+00	slug/ft^3
Modulus of Elasticity	6.4801000E+08	lb/ft^2
Poisson's Ratio	1.5000000E-01	
Thermal Expansion Coefficient	5.5000000E-06	1/F
Thermal Conductivity	1.5982900E-01	ft*lbs/(s*ft*°F)
High temp	1.2160000E+03	°R
Low Temp	4.9200000E+02	°R

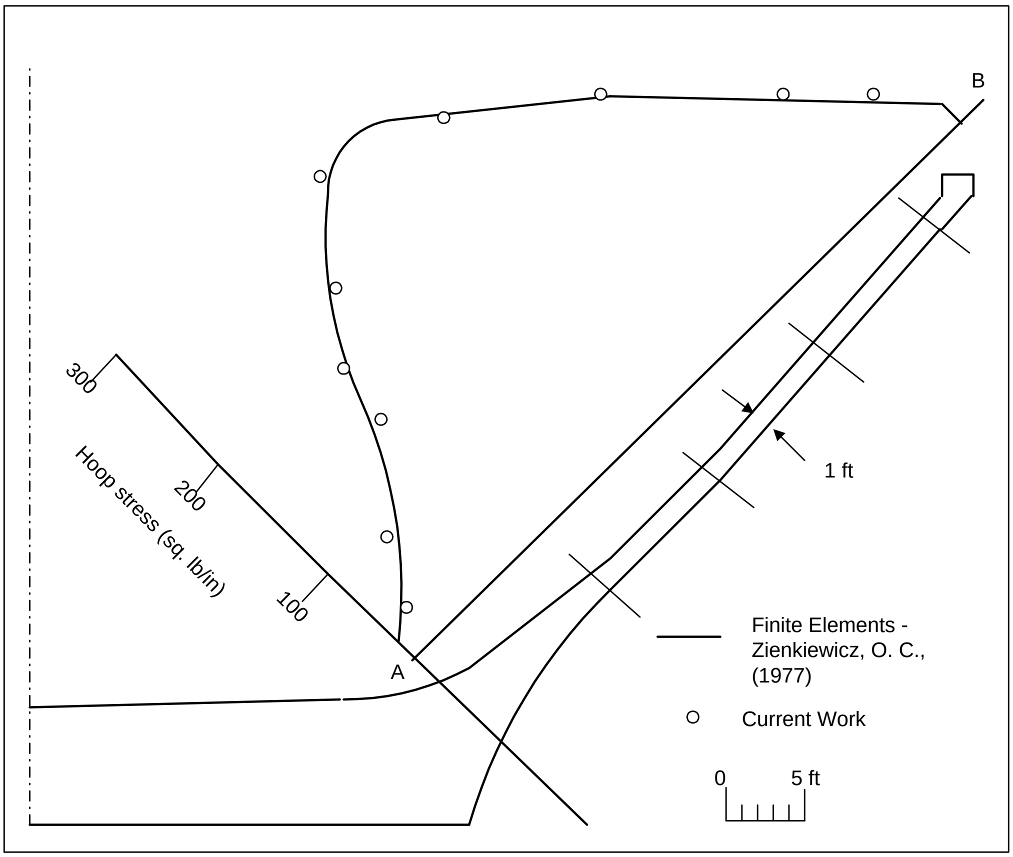

Figure 1. Conical Water Tank Subjected to Hydrostatic Pressure and Self-Weight

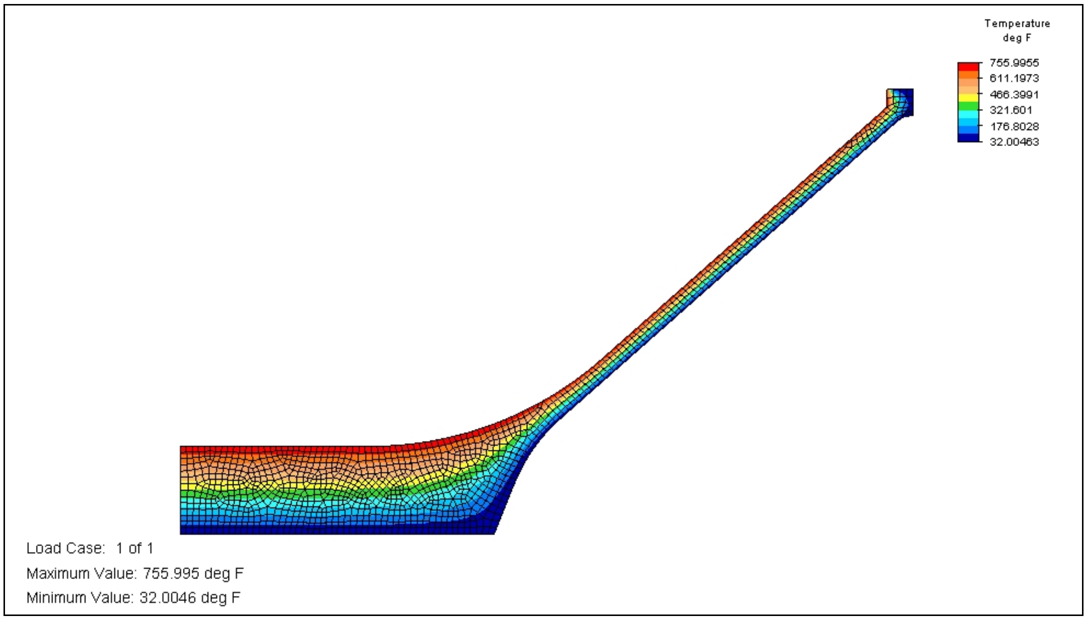

Figure 2: Steady state heat transfer results

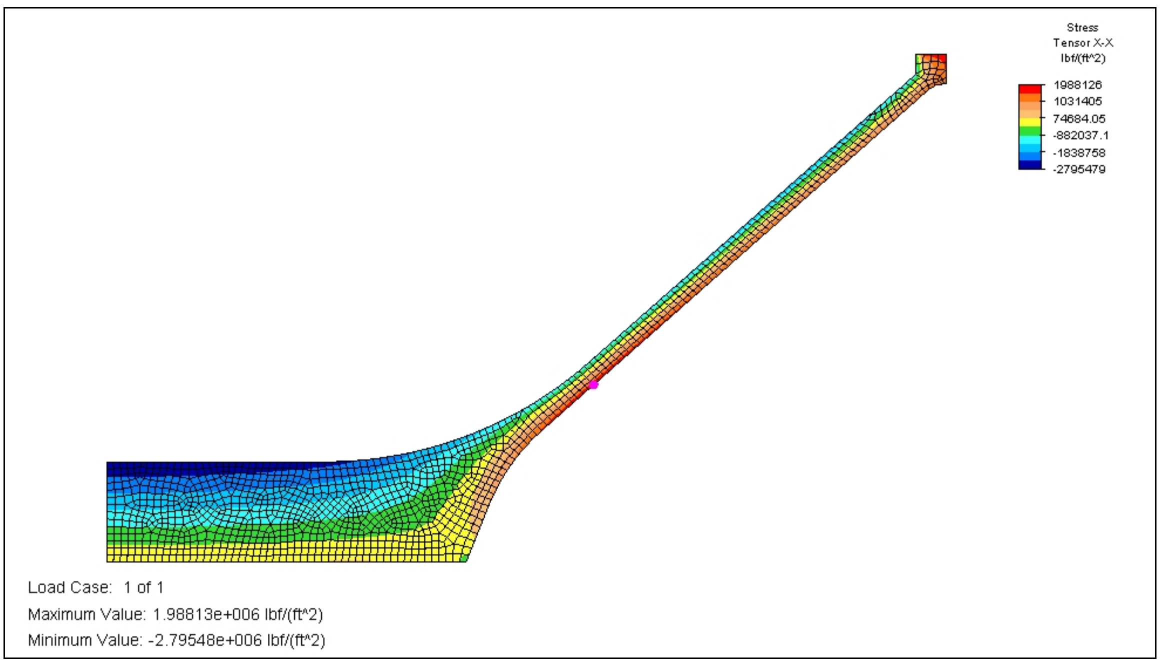

Figure 3: Linear static stress

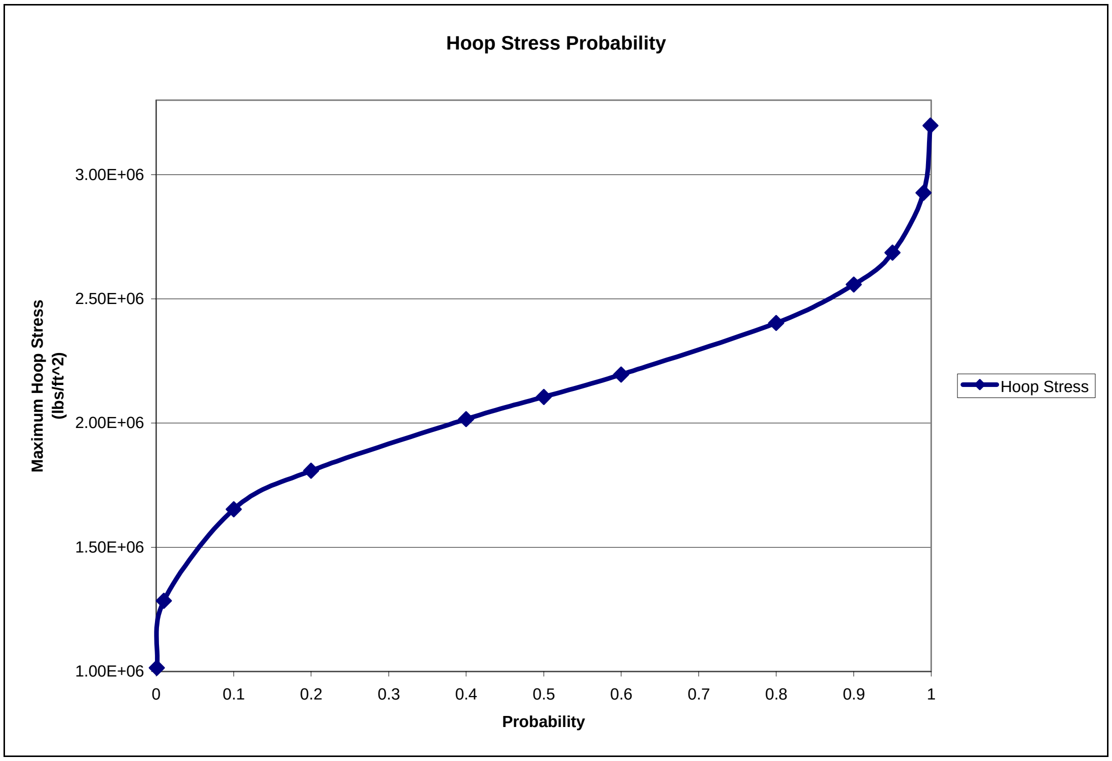

Figure 4 Cumulative Probability of Hoop Stress

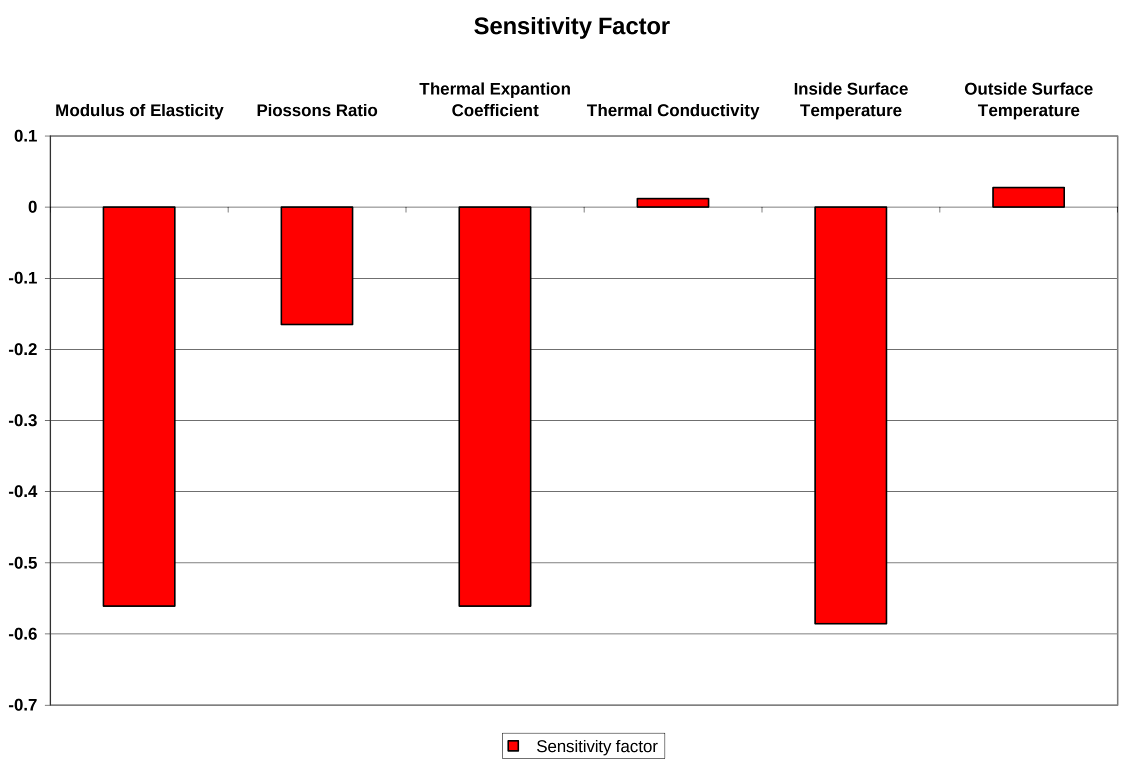

Figure 5. Sensitivity Factors versus Random Variables