Algorithmic Complexity Measure and Lyapanov matrices of the Dynamical Systems

Davoud Arasteh
Department of Electronic Engineering Technology
Southern University, Baton Rouge, LA 70813
davouda@engr.subr.edu

Abstract

The problem of distinguishing order from disorder in dynamical systems can be answered by certain quantities such as Lyapanov exponents, fractal dimensions, power spectrum density, and algorithmic complexity measures. In this paper, we have compared two approaches to evaluate the order and disorder in dynamic systems behavior. First, this is done by mapping the system output signal to a binary string and calculating the complexity measure of the time-series data. The results from algorithmic complexity are compared with the results from Lyapunov metrics computation. Using these two metrics, we can distinguish noise from chaos and order. This is important because modern engineering disciplines deal with signals acquired in the form of time series. The signals obtained from biological, electrical or mechanical systems appear to be complex. Therefore by extracting their characteristic features in such processes, one can make a correlation to a certain class of perception or behavior in cognitive sciences. This can be used for better analysis, control and diagnosis.

Introduction

In this paper we have addressed two practical approaches to evaluate the order and disorder in nonlinear systems output signal using algorithmic complexity measure [1,2,3] and largest Lyapanov exponent [4,5]. Mapping the system output signal to a binary string and calculating the complexity measure of the time-series data, does the characterization of strange attractors by the pattern formation in phase space attractors. The results obtained from complexity measure are compared with the results from Lyapunov metrics computation. In nonlinear chaotic systems small changes in initial conditions lead eventually to large changes in the behavior of the system. Nevertheless the system remains stable due to its deterministic nature, this occurs because the chaotic attractors with fractal geometry are confined in a certain region in phase space. For example in neural systems the divergence from initial state is the fundamental characteristics of perception to distinguish very close perceptual entities. The artificial cognitive informatics is concerned with the extraction of characteristic features, their measurement and characterization of phase space patterns in the processes related to perception and cognition. Signals obtained from such processes like EEG, ECG or behavioral signals appear to be random. Despite the fact that, these signals are not random and can be classified as chaotic [6,7,8,9,10]. There are several metrics to measure chaos, depending on what one wants to characterize in the chaotic trajectory. Certain quantities such as Lyapanov exponents, fractal dimensions, K-entropy, algorithmic complexity and power spectrum analysis have been used as screening tools to detect chaos in nonlinear systems. But methods like Fourier transform, and the resulting power spectrum density, fails to distinguish between chaos and noise, because both phenomena are broadband. This paper deals with the fundamental concept of measuring chaos in dynamical systems through algorithmic complexity measure and Lyapanov exponents. Algorithmic complexity is a useful practical tool to characterize spatiotemporal patterns of nonlinear dynamical systems. Both metrics are capable of distinguishing chaos from order but only complexity metric is capable of distinguishing deterministic chaos from random noise. In the next section, we introduce the concept of algorithmic complexity.

Mathematical background

Algorithmic complexity theory defines randomness based only on the characteristics of the signal, without any knowledge of the source of the data. Application of algorithmic complexity in multi-dimensional discrete and continuous dynamical systems as a characterizing parameter is discussed in [11]. We have used Henon discrete map, and forced-dissipative oscillator system to exemplify and illuminate the concept. We will see that for continuous nonlinear dynamical system algorithmic complexity measure is as powerful as other characteristics like spectrum of dimensions and entropies. The numerical effort needed to extract these spectra is rather large. This limits their determination to systems with dimensionality lower than ten. Therefore it is necessary to develop analytical tools in order to characterize chaotic motion in high-dimensional dynamical systems, e.g. spatiotemporal turbulence, or poorly stirred chemical reactions.

The algorithmic complexity of a string is defined to be the length in bits of the shortest algorithm required for a computer to produce the given string. For our purposes it is not necessary to assign an absolute value for the complexity of a string of bits. This means relative values are always sufficient. The shortest algorithms are referred to as minimal programs. The complexity of a string is thus the length in bits of the minimal program necessary to produce the given string. The definition of a random number can now be given as any binary string whose algorithmic complexity is judged to be essentially equal to the length of the string. Qualitatively, the information embodied in a random number cannot be reduced or compressed to a more compact form. As the string S grows in length, the length of program grows like n, and the length of the computer program in bits is essentially the same as the length of S. Such a string satisfies the definition of a random number since the algorithmic complexity of the string is essentially the same as the length in bits of the string.

One of the major challenges in chaotic dynamics is to extract a meaningful signal from data that have every appearance of being random. Clearly algorithmic complexity is a concept aimed specifically at the problem of distinguishing between the random and the nonrandom. Nevertheless it can distinguish between chaotic, quasi-periodic, and periodic signals. The problem lies in determining a computable measure of complexity. No absolute measure is possible because minimal programs by definition correspond to random numbers, and it is not possible to determine a truly random number in any formal system. Nevertheless, it is possible to define a measure of complexity. A relative measure is sufficient for many purposes. For the first time Kasper and Schuster applied this idea to dynamic systems exhibiting chaos based on work of Lempel and Ziv. The measure of complexity introduced by Lempel and Ziv is referred as LZ complexity for brevity. The LZ complexity measures the number of distinct patterns that must be copied to reproduce a given string. Therefore the only computer operations considered in constructing a string are copying old patterns and inserting new ones. Briefly described, a string S is scanned from left to right and complexity counter c (S) increased by one unit every time a new sub-string of consecutive digits is encountered in the scanning process. The resultant number c (S) is the complexity measure of the string S. Clearly any procedure such as this will over estimate the complexity of strings, but nevertheless we expect comparisons to be meaningful.

Our efforts here are directed toward outlining a computational algorithm and giving examples of the LZ complexity measure for various dynamic systems. In quantifying these ideas it becomes necessary to introduce certain definitions. Let A denote the alphabet of symbols from which the finite length sequences S are constructed and denote the length of these sequences as L (S) = n. A sequence S may be written in the form $S = S_1S_2S_3...S_n$, where S_i is mapped from phase space attractor pattern of a dynamic system based on a mapping rule. The vocabulary of a sequence S, denoted by V (S), is the set of all substrings of S. The LZ complexity of a given string S is the number of insertions of new symbols required to reconstruct S, where every attempt is made to construct S by copying alone without inserting any new symbols. The process is iterative and the first symbol must always be inserted. Notice that the minimum value for LZ complexity is two. Furthermore the LZ complexity measure for a given string S is unique and only relative values of c (n) are meaningful. In particular it is the comparison with the complexity of the random string that is meaningful. That is one should always compare the LZ complexity of a given string to the LZ complexity of random strings of the same length, $\lim_{n\to\infty} [c(n)/b(n)]$, where for a random string of length n, the LZ complexity is given by $b(n) = n/\log_2(n)$. Fig. 1 shows the complexity measure calculation flowchart.

Results

The usefulness of the algorithmic complexity measure as a characteristic metrics in dynamic systems to distinguish the order from disorder is studied next. In this approach, the phase space pattern of a certain nonlinear system is mapped to an array of symbols. Then the algorithmic complexity of the resulting bit string is calculated. Our first example is a famous discrete map called Henon map. The Henon map is a prototypical 2-D invertible iterated dynamical with chaotic solutions proposed by the French astronomer Michel Henon in 1976 [12]; as a simplified model of the Poincare map for the Lorenz model. In 1963 the meteorologist Edward Lorenz observed that a dynamical system with three coupled first-order nonlinear differential equations could lead to completely chaotic trajectories [13]. Non- linearity is a necessary, but not sufficient condition of chaos. It is necessary condition, because linear differential equations can be solved by Fourier transform procedures and do not lead to chaos. The system Lorenz used to model the dynamics of weather differs from Hamiltonian systems mainly by its dissipativity.

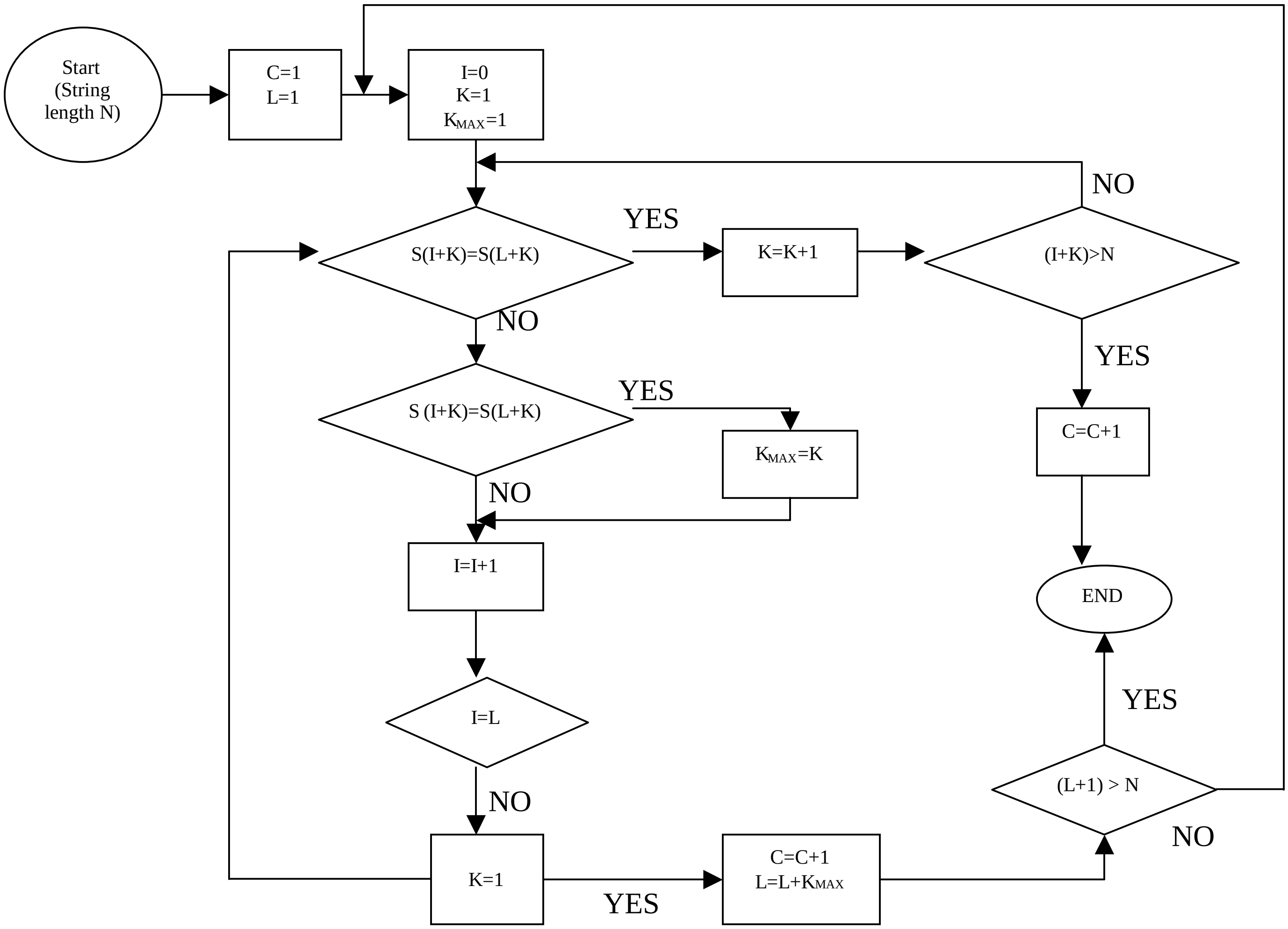

Figure 1 Flowchart for calculating LZ complexity for string S with length N

A dissipative system is not conservative but "open", with an external control parameter that can be tuned to critical values causing the transitions to chaos. Henon map can illustrate the basic concepts of complex dynamical systems from non-linearity to chaos with rather simple computer-assisted methods. Henon map is defined by following quadratic (nonlinear) recursive map: $X_{n+1} = 1 - a X_n^2 + Y_n$, $Y_{n+1} = b X_n$

Where $0<a<2$ and $|b|<1$. The parameter b is a measure of the rate of area contraction (dissipation), and the Henon map is the most general 2-D quadratic map with the property that the contraction is independent of x and y. For b = 0, the Henon map reduces to the quadratic map, which is conjugate to the logistic map $X_{n+1} = a X_n(1 - X_n)$. Bounded solutions exist for the Henon map over a range of "a" and "b" values, and a portion of this range yields chaotic solutions as shown in bifurcation diagram. For b= 0.3 and a_1=1 the sequence converges towards four fixed point. If "a" is increased beyond a critical value a_2=1.025, then the values of the sequence jump periodically between eight values after a certain time of transition (Fig. 2). If "a" is increased further beyond a critical value a_3=1.05, the period length doubles. If "a" is increased further and further; then the period doubles each time with a sequence of critical values a_2, a_3. But beyond a critical value a, the development becomes more and more irregular and chaotic. For these values, the mapping is contracting the area and has a trapping region, so it exhibits an attractor. However, for all values $|b|<1$, and for a wide range of values of a, the mapping is still contracting the area, and still has a trapping region. The sequence of period doubling bifurcations and chaotic region is illustrated in Fig. 1. We investigated the existence and transformation of the chaotic attractor of the Henon mapping with numerical methods. We used two characteristic metrics to identify these regions. We examine the complexity of the Henon map by establishing an alphabet and a scheme for constructing strings from this alphabet. The Henon map phase space is divided into four regions: (-X,-Y), (X,Y), (-X,Y), (X, -Y). When the dynamical response of the system (X_n ,Y_n) for a given control parameter, relies in first and third quadrant of phase space; digit "1" is mapped to the string, otherwise digit "0" is inserted. The source entropy for this mapping scheme, $h = -[p \log_2 p + (1-p) \log_2 (1-p)]$, where p is the probability that the dynamical response lies in the first and third quadrant of the phase space is calculated. We found that for Henon chaotic attractor source entropy is close to unity, which justifies our choice of mapping scheme. We regard 10,000 points on the attractor as sufficient for measuring the complexity of two-dimensional attractors generated by Henon map. Each array with length n =10000, corresponds to 10000 iterations stored after transient values are ignored. Each experiment is performed for a specific value of control parameters. Parameter "a" is varied in the range [1, 4] increased by step size Δa = 0.001, and "b" is kept constant at 0.3 during the simulation. Fig. 2 shows the LZ complexity of Henon map versus control parameter "a". Furthermore bifurcation diagram is plotted for the same range of values of parameter "a". It is clear that when Henon system has periodic behaviors of 1-cycle, 2-cycle… that is inside bifurcation windows, the complexity measure is small compared to the situations when system behavior is chaotic. The results indicate distinct regions corresponding to chaotic and periodic behaviors. For values above a =1.23 and less than 1.26, c (n) shows a wide minima at the windows, as expected from bifurcation diagram and reaches again its largest possible value at a =1.4 where the Lyapanov exponent of the Henon map has also its maximum. This means that two different kinds of behavior of c (n) have been found for the Henon map. For periodic orbits, c (n) reaches a finite value for large n (at the theoretical limit $n\rightarrow\infty$, it goes to zero). At period doubling windows, c (n) diverges with n, but the normalized complexity c (n)/b (n), i.e., the complexity per digit approaches to zero. For chaotic orbits the normalized complexity c (n)/b (n) asymptotically reaches a finite positive value less than one. For example at a =1.4, c (n)/b (n) is 0.74. This means that the complexity c is a more precise measure than the Lyapanov exponent for characterizing order or disorder. Our results are shown through bifurcation diagram, algorithmic complexity and Lyapunov exponent.

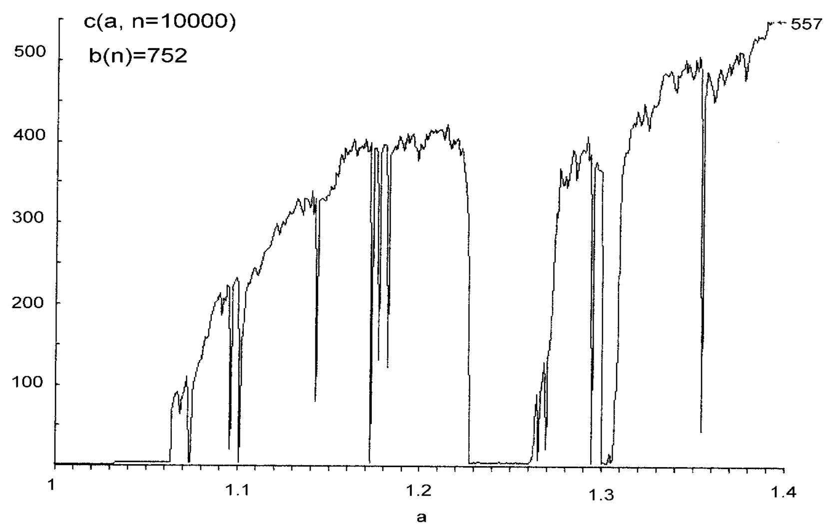

Figure 2 The LZ complexity of strings constructed from the Henon map as a function of control parameter "a"

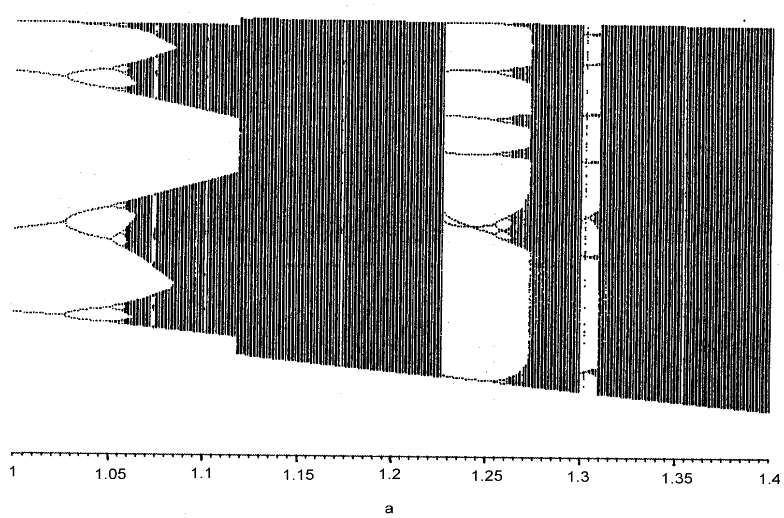

Figure 3 Henon bifurcation diagram vs. control parameter "a".

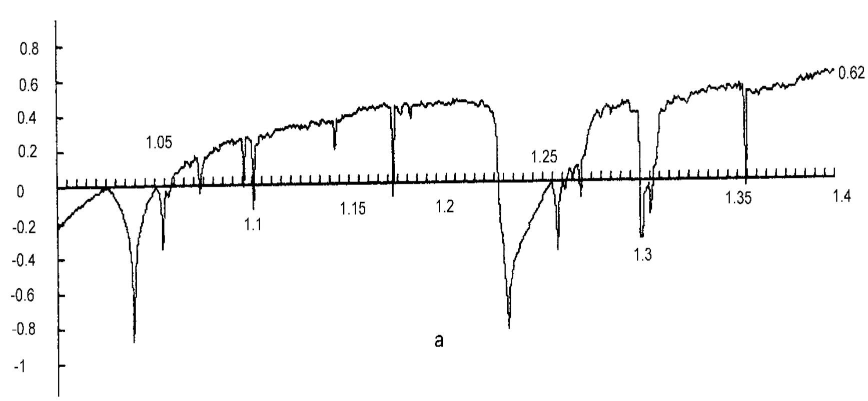

Figure 4 Largest Lyapunov exponent as a function of control parameter "a"

Our next example is a continuous dynamical system called Forced-Dissipative-Oscillator. It is used as a mathematical model for many physical and engineering systems like forced-dissipative pendulum and Josephson junction under microwave radiation [14, 15, 16], given by following dynamical equation: $D^2 X + k(DX) + \sin X = g \cos(\omega_d t)$, $DX = dX/dt$

Control parameters are; "k" as system dissipative coefficient, "g" as external driving force amplitude, and ω_d external driving force frequency. Roughly speaking, a dissipative system is not conservative but "open", with an external control parameter that can be tuned to critical values causing the transitions to chaos. More precisely, conservative as well as dissipative systems are characterized by nonlinear differential equations $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}, \mathbf{G})$ with a nonlinear function $\mathbf{F}$ of the vector $\mathbf{x} = (x_1, ..., x_d)$ depending on an external control parameter $\mathbf{G}$. While for conservative systems the volume elements in the corresponding phase space change their shape but retain their volume in the course of time, the volume elements of dissipative systems shrink as time increases. In dissipative system, or non-Hamiltonian systems, the area-preserving principle does not apply. In fact, the sum of all the Lyapunov exponents must be negative for physical dissipative dynamical system. A one-dimensional map like logistic map there is only one Lyapunov exponent. If it is negative, the map has either limit point stability or limit cycle stability. In n-dimensional systems, the stretching and contraction along the principal axis in the phase space produce a spectrum of Lyapunov exponents. For example in three-dimension the Lyapunov spectrum is ($\lambda 1$, $\lambda 2$, $\lambda 3$). Stable periodic attractors have only zero and negative Lyapunov exponents. For example, ($\lambda 1 = 0$, $\lambda 2 =$ negative number, $\lambda 3 =$ negative number) where the zero corresponds to the limit-cycle trajectory itself or ($\lambda 1 = 0$, $\lambda 2 = 0$, $\lambda 3 =$ negative number) for an attracting 2-torus. Chaotic attractors have just one finite positive Lyapunov exponent. In three-dimension systems the spectrum of the Lyapunov exponents is (+, 0, -), where the zero corresponds to the chaotic trajectory itself, with some trajectories expanding, while others are contracting. For forced oscillator system we have used both tools to explore the regions of periodic and chaotic behaviors.

The complexity of the oscillator map is calculated by setting control parameters to following values. $\omega_d = 2/3$, $k = 0.5$, and g is varied in the range [0.9, 1.5], with $\Delta g = 0.001$. An array from n= 20000 iterations is constructed after removing the transient values for a specific external force amplitude (g). A digit "1" is mapped into the array if the velocity (DX) of the oscillator is larger than zero (a pendulum passing its equilibrium state), otherwise digit "0" is inserted in the array. The choice of using this encoding scheme is based on the symmetrical velocity distribution function. This function is calculated from distribution density of 80000 phase space points of a chaotic attractor (Fig. 5), with normalized frequency of 2/3, dissipation coefficient of 0.5, and external driving amplitude of 1.2. We have computed the algorithmic complexity measure of each experiment versus the corresponding driving force amplitudes (Fig. 6). For comparison we have also plotted the bifurcation diagram (Fig. 7) as well as the largest Lyapunov exponent (Fig. 8). The results indicate distinct regions corresponding to chaotic and periodic behaviors. When Forced-Dissipative-Oscillator system has periodic behaviors of 1-cycle, 2-cycle, … and displayed in bifurcation windows, the LZ complexity measure is small. This corresponds to the limit-cycle trajectory or an attracting 2-torus. On the other hand chaotic attractors with some trajectories expanding, while others are contracting, have large LZ complexity, this is nicely mirrored by the increasing values of c (n) plot. For this system there is only one finite positive Lyapanov exponent and only one symbolic Lyapanov spectrum of the form (+, 0, -), where the zero corresponds to the chaotic trajectory itself, with some trajectories expanding, while others are contracting.

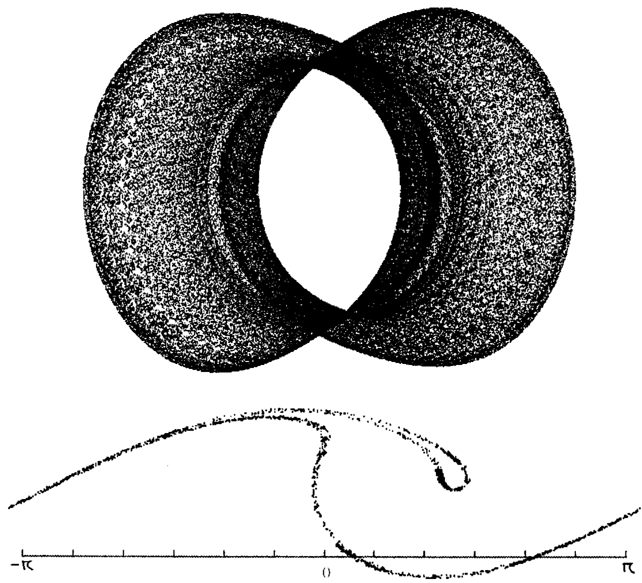

Figure 5 Forced-dissipative oscillator strange attractor and its Poincare map $g=1.2$, $k=0.5$, $\omega_d=2/3$

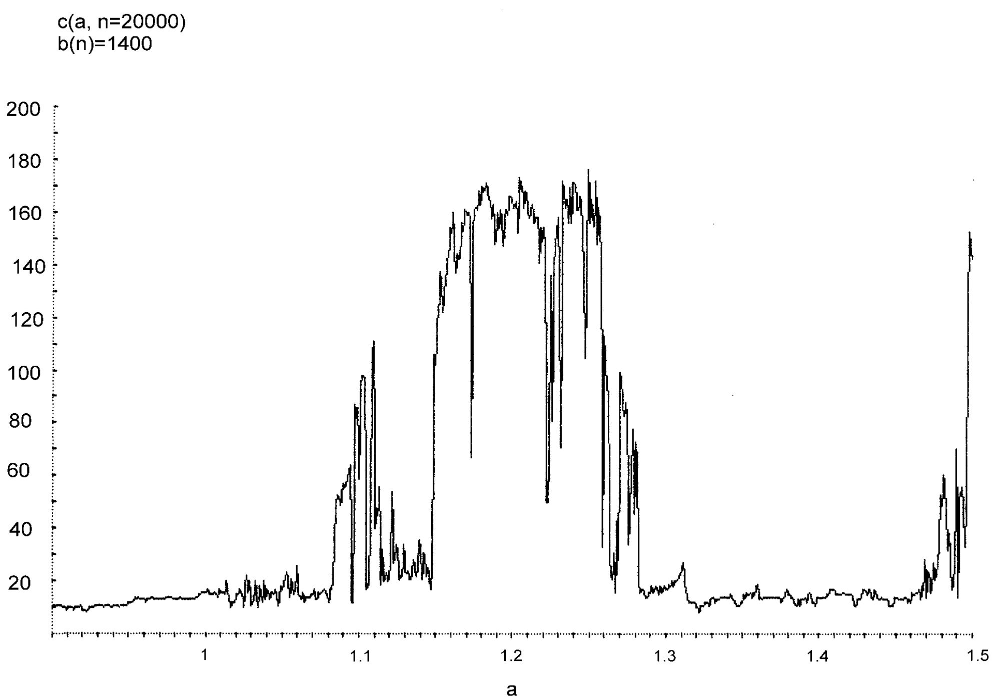

Figure 6 Forced-Dissipative-Oscillator algorithmic complexity measure versus control parameter g, amplitude of the external force $(0.9<g<1.5)$, $k=0.5$, $\omega_d = 2/3$

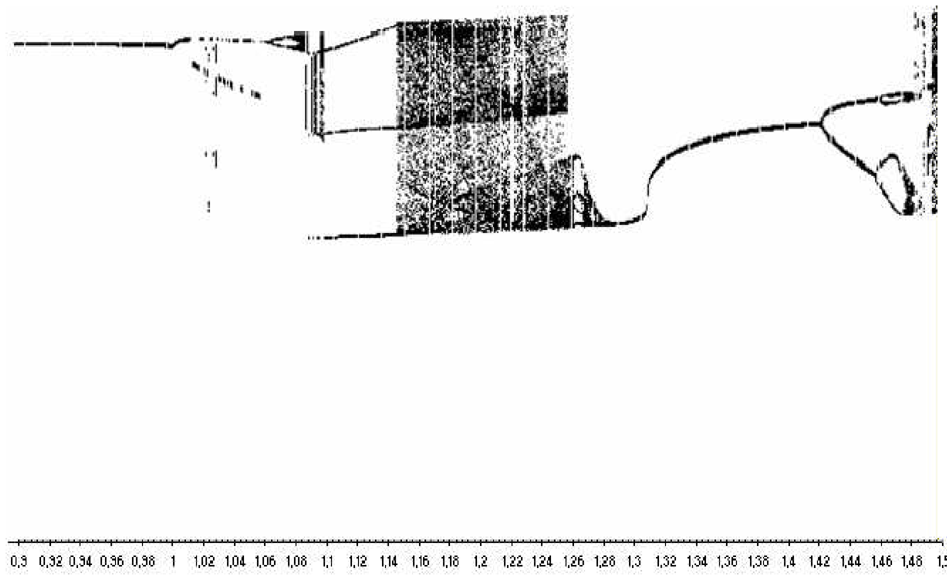

Figure 7 FDO bifurcation diagram versus control parameter g, amplitude of the external force (0.9<g<1.5), k=0.5, ω_d=2/3

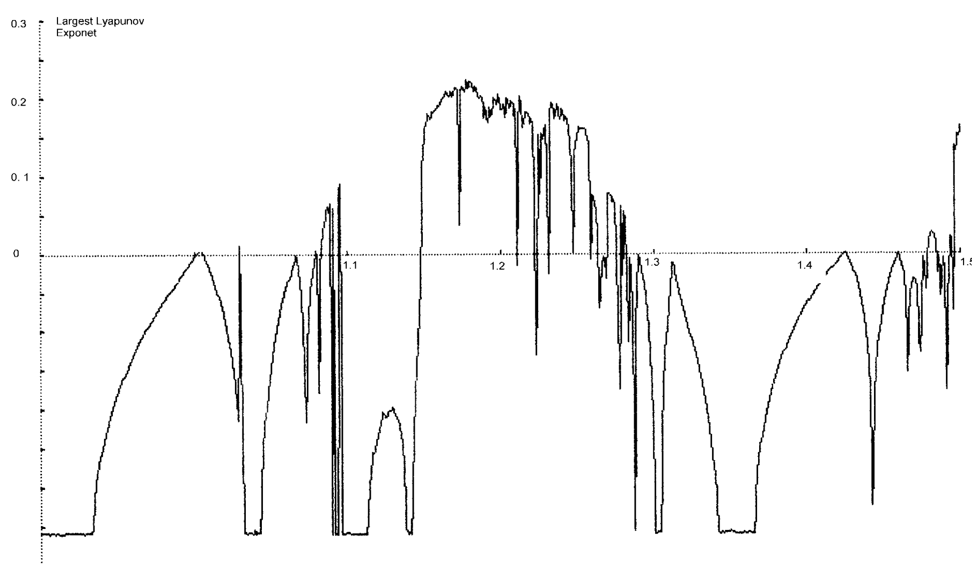

Figure 8 Forced-Dissipative-Oscillator largest Lyapunov exponents vs. external force amplitude (0.9<g<1.5) k=0.5, ω_d=2/3

Conclusion

We investigated the whole set of control parameter values that generate chaotic attractors with a new exploration tool. To characterize their responses for the different values of control parameters we have computed the algorithmic complexity measure of the phase space attractors. Henon discrete map and forced dissipative oscillator dynamic systems were chosen for numerical experiment. We observed that the algorithmic complexity of the chaotic responses is much higher than periodic responses. In this work we showed the usefulness of LZ complexity measure as a metric to characterize the patterns of discrete and continuous dynamical systems with chaotic and periodic responses. This is important because it provides a computational metric to find and classify the complex patterns.

References

[1] A. Lempel, J. Ziv, "On the complexity of individual sequences", IEEE Transaction in Information Theory Vol. IT-22, p. 75, 1976.

[2] A. Lempel, J. Ziv, "Compression of Individual Sequences via Variable-Rate Coding", IEEE Transaction in Information Theory, Vol. IT-24, NO. 5, 1978.

[3] Torres ME and L. G. Gamero, "Relative Complexity Changes in Time Series using Information Measures", Physica A, Vol. 286, Iss. 3-4, pp.457-473, Oct. 2000.

[4] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov exponents from a time series", Physica D, vol. 16, pp. 285–317, 1985.

[5] W. Kinsner, "Characterizing Chaos Through Lyapunov Metrics", IEEE Transactions on Systems, Man, and Cybernetics, Part C, Vol. 36, No. 2, 2006.

[6] Szczepanski J, JM Amigo, E Wajnryb and MV Sanchez-Vives, "Application of Lempel–Ziv complexity to the analysis of neural discharges", Network: Computation in Neural Systems, 14 pp. 335–350, 2003.

[7] Gonzalez Andino SL, Grave de Peralta Menendez R, Thut G, Spinelli L, Blanke O, Michel CM, Seeck M, Landis T., "Measuring the complexity of time series: an application to neurophysiological signals". Hum Brain Mapp. 2000 Sep;11(1):46-57.

[8] Watanabe TA, Cellucci CJ, Kohegyi E, Bashore TR, Josiassen RC, Greenbaun NN, Rapp PE. "The algorithmic complexity of multichannel EEGs is sensitive to changes in behavior" Psychophysiology, Jan;40(1):77-97, 2003

[9] Jaeseung, J., Jeong-Ho C., Kim, S. Y. & Seol-Heui, H., "Nonlinear dynamical analysis of the EEG in patients with Alzheimer's disease and vacular dementia", Clin., Neurophysiol 18(1):58-67, 2001.

[10] Huang L, Fengchi Ju, Enke Zhang, Jingzhi Cheng, "Real-time Estimation of Depth of Anaesthesia Using the Mutual Information of Electroencephalograms", Proceedings of the 1st International IEEE EMBS Conference on Neural Engineering Capri Island. Italy March 20-22, 2003.

[11] F. Kasper, H.G. Shuster, "Easily calculable measure for the complexity of spatiotemporal patterns." Physical Review A, Vol. 36, p. 832, 1987.

[12] M. Henon, "A two-dimensional mapping with a strange attractor", Comm. Math. Phys. Vol. 50, p. 69, 1976.

[13] E. N. Lorentz, ”Deterministic Nonperiodic Flow”, J. Atoms Sci., Vol. 20, p.130, 1963. In “Universality in Chaos”, edited by P. Civitanovic, Adam Hilger, 1989.

[14] C. B. Whan and C. J. Lobb, “Complex Dynamical Behavior in RLC-shunted Josephson Junctions”, IEEE Transaction on Superconductivity, Vol. 5, No. 2, June 1995.

[15] D. D’ Humierres, M. R. Beasley, B. A. Libchaber, “Chaotic states and routes to chaos in the forced pendulum”, Physical Review A, Vol. 26, No. 6, p. 3483, 1982.

[16] B.A. Huberman, J. P. Crutchfield, N. H. Packard, “Noise phenomena in Josephson Junction”, Applied Physics Letter, Vol. 37, 750, 1980.

Biography

Davoud Arasteh serves as an assistant professor of Electronic Engineering Technology at Southern University of Baton Rouge. His research interests include Mobile Computing, Network Security, Nonlinear Dynamical Systems, Computer Vision, and Technology Based Engineering Education. He is the chair of departmental curriculum committee and is a member of ASEE, IEEE, IEEE Computer Society, IEEE Electromagnetic Compatibility Society, and IEEE Computational Intelligence Society.