Quickest Descent Control of Nonlinear Systems with Singular Control Chatter Elimination

Dale B. McDonald
McCoy School of Engineering
Midwestern State University
dale.mcdonald@mwsu.edu

Abstract

In this treatment, the control of a class of nonlinear mechanical systems is considered. The objective is to zero the states of the system with a bounded control, while considering some state dependent cost. Specification of controls for this system class is extremely difficult due to the nonlinearities and bounds on the control. Furthermore, such controls often produce chatter; undesirable behavior that can damage physical components through vibration and excitement of resonant modes. A control law is proposed that minimizes cost in a quickest descent fashion; that is, at each instant a control is chosen that minimizes the time rate of change of the cost function. The control law includes a singular control regime that alleviates the chattering issue and provides asymptotic stability of the origin. This singular regime produces a switching surface, whose existence and placement is driven by function minimization in addition to stability of reduced order dynamics. It is shown that the control law provides the stability result when traditional Lyapunov optimizing controls cannot; therefore the applicability of such schemes is extended for this problem class.

Introduction

Lyapunov's second method has been used extensively to develop control laws for linear and nonlinear systems, for example as in [1,3,4,5]. One such method, Lyapunov optimizing control (LOC), produces feedback controls by selecting a candidate Lyapunov function and choosing the control to minimize this function as much as possible along system trajectories [2, 8, 10]. The primary advantage that is realized during the use of LOC algorithms comes from a built in proof of their effectiveness. Assuming the target is the origin, if the candidate Lyapunov function is decreased everywhere outside the origin, a sufficient condition for asymptotic stability is satisfied [9]. Therefore, the LOC approach provides a means to design feedback controls where cost accumulation, stability of the origin, and control bounds are explicitly considered. The difficulty in utilizing LOC methods arises during proof of asymptotic stability of the origin. For a dynamical system it may be quite difficult to guarantee that the candidate Lyapunov function decreases everywhere outside the origin. If the function doesn't decrease everywhere apart from the origin, asymptotic stability using traditional means cannot be proven. A second difficulty arises that is notorious for the problem class considered here; the so-called chatter phenomenon. Chatter is typified by high frequency (discontinuous) commutation of the control signal across a switching surface [7, 11]. This commutation may produce a variety of detrimental effects on a mechanical system such as excitement of resonant modes and extreme wear on actuators. In this treatment, we use a variant of LOC known as quickest descent control (QDC) to derive a control algorithm that provides desired asymptotic stability of the origin, considers cost and control bounds explicitly, and uses singular control to eliminate the discontinuous chatter. These objectives are achieved without requiring the time rate of change of the candidate Lyapunov function to be strictly negative outside the origin.

Problem Formulation

This research considers the optimal control problem

$$J = \int_{0}^{t_f} \Psi(\mathbf{x}) dt \qquad (1)$$

where

$$\Psi(\mathbf{x}) = \frac{1}{2} \mathbf{x}^{\mathbf{T}} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} \mathbf{x} \qquad (2)$$

which is associated with transporting the state $\mathbf{x}$ from some initial point to the origin. The matrix $\partial^2 \Psi / \partial \mathbf{x}^{\mathbf{T}}$ is constant, symmetric, and positive definite. We consider nonlinear second-order systems

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) + u \end{aligned} \qquad (3)$$

where the control variable u is an element of the constraint set

$$u_{\min} \le u \le u_{\max} \qquad (4)$$

where $u_{\min} = -u_{\max}$. Furthermore, we assume to know a function $\overline{f}$ that satisfies

$$\left| f(x_1, x_2) \right| < \overline{f}\left(\|\mathbf{x}\| \right)$$

$$\left\| \overline{f}(\mathbf{x}) \right\| \to 0 \; as \; \|\mathbf{x}\| \to 0 \qquad (5)$$

Quickest Descent Control

Quickest descent control [6, 8] provides the structure for algorithms developed in this treatment that consider cost accumulation, control bounds, and progress to the target. Controls found by the application of QDC are derived by first selecting a descent function $\overline{\Psi}(\mathbf{x})$ that exhibits the following properties: 1) $\overline{\Psi}(\mathbf{0}) = 0$, 2) $\overline{\Psi}(\mathbf{x}) > 0 \;\forall\; \mathbf{x} \ne \mathbf{0}$, and 3) $\partial \overline{\Psi} / \partial \mathbf{x} \ne \mathbf{0} \;\forall\; \mathbf{x} \ne \mathbf{0}$. The control u for the problem (1)-(4) is chosen to decrease $\overline{\Psi}(\mathbf{x})$ as quickly as possible along trajectories $\mathbf{x}$(t). For that reason, u is chosen by

$$\min_u \frac{d\overline{\Psi}}{dt} \qquad (6)$$

where we choose the cost accumulation rate $\Psi(\mathbf{x})$ as the descent function; that is we let $\Psi(\mathbf{x}) = \overline{\Psi}(\mathbf{x})$. Therefore, the control is chosen

$$\min_u \dot{\Psi} \qquad (7)$$

where $\dot{\Psi} = \mathrm{d}\Psi/\mathrm{dt}$. If $\dot{\Psi} < 0 \forall \mathbf{x} \ne 0$ and properties (1) - (3) are satisfied, then we are assured that the origin is asymptotically stable. In [5] necessary conditions are formulated on $\Psi(\mathbf{x})$ that allow for stability of the target. Here we assume these conditions are satisfied so that we may prove the desired stability results of the chattering-free controller.

Properties of the Controller

Given a satisfactory $\Psi(\mathbf{x})$ we have

$$\dot{\Psi}(\mathbf{x},u) = (\partial\Psi/\partial x_1)x_2 + (\partial\Psi/\partial x_2)[f(x_1,x_2)+u] \tag{8}$$

Application of (7) results in the following Quickest Descent Control Law (**QDCL**)

u(x) =	$u_{\max}$	if $\sigma(\mathbf{x}) < 0$
	u_s	if $\sigma(\mathbf{x}) = 0$
	$u_{\min}$	if $\sigma(\mathbf{x}) > 0$

where singular control [8] is denoted by u_s. Furthermore,

$$\sigma(\mathbf{x}) = \partial\Psi/\partial x_2 \tag{9}$$

with switching surface $S \equiv \{x : \sigma(x) = 0\}$. Additional entities related to the switching surface are

$$\partial\sigma/\partial\mathbf{x} = \partial^2\Psi/\partial x_2\partial\mathbf{x} \tag{10}$$

$$\dot{\sigma} = \frac{\partial^2\Psi}{\partial x_2\partial x_1}(x_2) + \frac{\partial^2\Psi}{\partial x_2^{\ 2}}[f(x_1,x_2)+u] \tag{11}$$

Singular control u_s requires $\sigma = \dot{\sigma} = 0$, which from (9)-(11), is

$$u_s(\mathbf{x}) = -\frac{\frac{\partial^2\Psi}{\partial x_2\partial x_1}(x_2)}{\frac{\partial^2\Psi}{\partial x_2^{\ 2}}} - f(x_1,x_2). \tag{12}$$

Stability Results

The proposed control law **QDCL** is useful only if it can be shown that for the nonlinear system (3) subject to control bounds (4), there exists a region in state space from which trajectories reach S and then asymptotically approach the origin.

1. Reaching Phase

To prove asymptotic stability of the origin, we first prove that there exist initial states whose trajectories, under **QDCL**, reach S in finite time.

Theorem 1. Assume the necessary conditions of [5] are satisfied by

$$\Psi(\mathbf{x}) = \frac{1}{2}kx_1^{\ 2} + bx_1x_2 + \frac{1}{2}mx_2^{\ 2} \tag{13}$$

note that this implies that the constants k, b and m are strictly positive. With these necessary conditions satisfied, the function (13) can be seen as penalizing states that are not at the origin (a positive cost is imposed, which we desire to minimize). Over any finite time interval beginning at $t = t_0$, the state may be upper bounded by a function of $\|\mathbf{x}(t_0)\|$ and by the maximum absolute value of σ attained in that time interval.

Proof. Consider the first component of the state equations (3)

$$\dot{x}_1 = x_2 = -\frac{b}{m}x_1 + \frac{1}{m}\sigma \tag{14}$$

therefore

$$x_1(t)=e^{-\frac{b}{m}(t-t_0)}x_1(t_0)+\frac{1}{m}\int_{t_0}^{t}e^{-\frac{b}{m}(t-\tau)}\sigma(\tau)d\tau \tag{15}$$

$$\leq e^{-\frac{b}{m}(t-t_0)}x_1(t_0)+\left(\frac{1}{m}\right)\max_{t_0\leq\tau\leq t}\sigma(\tau)\int_{t_0}^{t}e^{-\frac{b}{m}(t-\tau)}d\tau \tag{16}$$

note that the integrals may be evaluated since (16) is a stable system with *b* and *m* positive. Now, $x_1(t)$ satisfies

$$|x_1(t)|\leq|x_1(t_0)|+\left(\frac{1}{m}\right)\max_{t_0\leq\tau\leq t}\sigma(\tau)\int_{t_0}^{\infty}\left|e^{-\frac{b}{m}(t-\tau)}\right|d\tau \tag{17}$$

and

$$\begin{aligned}|x_2(t)|&=\left|-\frac{b}{m}x_1+\frac{1}{m}\sigma\right|\\&\leq\left|\frac{b}{m}x_1\right|+\frac{1}{m}\max_{t_0\leq\tau\leq t}|\sigma|\\&\leq\frac{b}{m}\left[|x_1(t_0)|+\left(\frac{1}{m}\right)\max_{t_0\leq\tau\leq t}\sigma(\tau)\int_{t_0}^{\infty}\left|e^{-\frac{b}{m}(t-\tau)}\right|d\tau\right]+\max_{t_0\leq\tau\leq t}\frac{|\sigma(\tau)|}{m}\end{aligned} \tag{18}$$

and therefore (noting that $\|\mathbf{x}\|\leq|x_1(t)|+|x_2(t)|$ and using (17))

$$\|\mathbf{x}(t)\|\leq\left(1+\frac{b}{m}\right)\left[|x_1(t_0)|+\left(\frac{1}{m}\right)\max_{t_0\leq\tau\leq t}\sigma(\tau)\int_{t_0}^{\infty}\left|e^{-\frac{b}{m}(t-\tau)}\right|d\tau\right]+\max_{t_0\leq\tau\leq t}\frac{|\sigma(\tau)|}{m} \tag{19}$$

which completes the proof.
Leveraging the bound (19) on the state, we can prove finite time convergence to *S*.

Theorem 2. Consider the region $\sigma(\mathbf{x})<0$; all $\mathbf{x}(t_0)$ that satisfy

$$\frac{\partial\sigma}{\partial x_2}u_{\max}>\frac{\partial\sigma}{\partial x_1}\left[\frac{b}{m}\left(|x_1(t_0)|+\left(\frac{1}{m}\right)\max_{t_0\leq\tau\leq t}\sigma(\tau)\int_{t_0}^{\infty}\left|e^{-\frac{b}{m}(t-\tau)}\right|d\tau\right)+\max_{t_0\leq\tau\leq t}\left|\frac{\sigma(\tau)}{m}\right|+\frac{\partial\sigma}{\partial x_2}\bar{f}(\|\mathbf{x}\|)\right] \tag{20}$$

reach the surface S in finite time.

Proof. First, we are assured such a region exists; for arbitrarily small $\mathbf{x}(t_0)$ and $\sigma(t_0)$ since

$$\frac{\partial\sigma}{\partial x_2}u_{\max}>0.$$

Note that (20) implies that $\dot{\sigma}>0$. To prove finite time convergence we must show that $\dot{\sigma}(t)$ is strictly positive for all $\mathbf{x}(t_0)$ that satisfy (20). The proof follows by supposing for some $\mathbf{x}(t_0)$ there is a time when $\dot{\sigma}(t)=0$. The first instant at which this is true requires, recalling (11)

$$0=\frac{\partial\sigma}{\partial x_1}x_2+\frac{\partial\sigma}{\partial x_2}f(x_1,x_2)+\frac{\partial\sigma}{\partial x_2}u_{\max}$$

where $u=u_{max}$ since $\sigma<0$. This implies that

$$\frac{\partial \sigma}{\partial x_2} u_{\max} = -\left[\frac{\partial \sigma}{\partial x_1} x_2 + \frac{\partial \sigma}{\partial x_2} f(x_1, x_2) \right]$$

and with $(\partial \sigma / \partial x_2) u_{\max} > 0$, $\dot{\sigma} = 0$ requires

$$\begin{aligned} \frac{\partial \sigma}{\partial x_2} u_{\max} &= \left\| -\left[\frac{\partial \sigma}{\partial x_1} x_2 + \frac{\partial \sigma}{\partial x_2} f(x_1, x_2) \right] \right\| \\ &\leq \left\| \frac{\partial \sigma}{\partial x_1} x_2 \right\| + \left\| \frac{\partial \sigma}{\partial x_2} f(x_1, x_2) \right\| \\ &\leq \frac{\partial \sigma}{\partial x_1} \| x_2 \| + \frac{\partial \sigma}{\partial x_2} \left\| \bar{f}(\| \mathbf{x} \|) \right\| \end{aligned}$$

resulting in (using (18))

$$\frac{\partial \sigma}{\partial x_2} u_{\max} \leq \frac{\partial \sigma}{\partial x_1} \left[\frac{b}{m} \left(\left| x_1(t_0) \right| + \left(\frac{1}{m} \right) \max_{t_0 \leq \tau \leq t} \sigma(\tau) \int_{t_0}^{\infty} \left| e^{-\frac{b}{m}(t-\tau)} \right| d\tau \right) + \max_{t_0 \leq \tau \leq t} \left| \frac{\sigma(\tau)}{m} \right| + \frac{\partial \sigma}{\partial x_2} \bar{f}(\| \mathbf{x} \|) \right]$$

which is a contradiction. Therefore, trajectories satisfying (20) reach *S* in finite time. Analogous results hold for $\sigma > 0$.

2. Asymptotic Stability

Theorem 3. With constants *m* and *b* greater than zero, trajectories that reach S with u_s satisfying (4) asymptotically approach the origin.

Proof. The proof follows by substitution of (12) into (3) yielding

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{\dfrac{\partial^2 \Psi}{\partial x_2 \partial x_1}}{\dfrac{\partial^2 \Psi}{\partial x_2{}^2}} \end{aligned} \tag{21}$$

With $\left(\partial^2 \Psi / \partial x_2 \partial x_1 \right) = b$ and $\left(\partial^2 \Psi / \partial x_2{}^2 \right) = m$, (21) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m} x_2 \end{aligned} \tag{22}$$

Clearly $x_2 \to 0$. With u_s enforcing the constraint ($\sigma = bx_1 + mx_2 = 0$) and $x_2 \to 0$, $x_1 \to 0$ and the proof is complete. Note that in the proofs of the reaching and stability phases of the control law we did not require $\dot{\Psi} < 0$ for $\mathbf{x} \neq \mathbf{0}$. This is the advantage of the proposed control law **QDCL** over traditional Lyapunov optimizing control schemes.

Numerical Simulation

To demonstrate the usefulness of the proposed control scheme, we present analysis and simulation results for a representative nonlinear system. In the present case, problem is the stabilization of a nonlinear inverted pendulum about an unstable (vertical) equilibrium position. The control objective was to stabilize the origin, while minimizing total energy with dissipation (13) with $m = 4$, $b = 2$, and $k = 3$. The state equations took the form of (3), with $f(x_1, x_2) = \sin(x_1) - x_2$ and we assumed that $u_{max} = 10$. This choice of $u_{max} = 10$ is dependent upon the hardware capabilities of the control system. For example, an inverted pendulum is typically controlled by an electric motor; the maximum torque delivered by this motor is reflected in the maximum control effort ($u_{max} = 10$).

First, the control law **QDCL** was implemented, but singular control was not used; Across the switching surface, the control varied discontinuously between its minimum and maximum values.

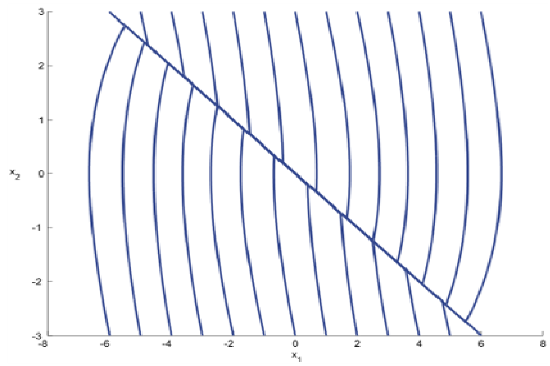

Figure 1: Typical chattering trajectories without singular control.

In Fig. 1, a wide view is shown; the trajectories asymptotically approach the origin. However, in Fig. 2, which is a close-up of a portion of Fig. 1, we observe the chatter that presents due to the discontinuous control law. This chatter can cause wearing of mechanical components and may induce resonant modes; it should be avoided whenever possible.

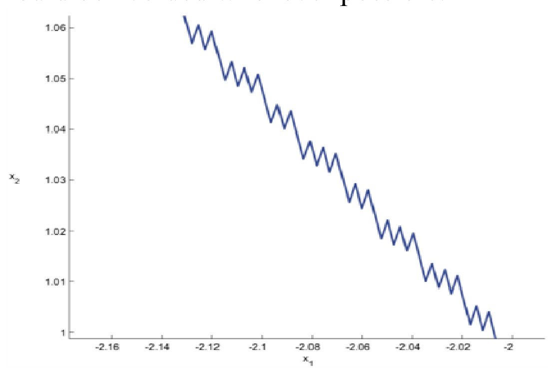

Figure 2: A Typical chattering trajectory without singular control.

In Fig. 3, typical trajectories are shown as a result of implementing the control law **QDCL** (including singular control). Use of the singular control regime eliminates the discontinuous chatter.

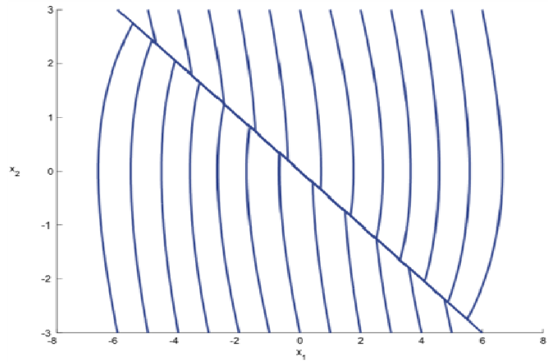

Figure 3: Typical trajectories with singular control (CLO)

In Fig. 4, a close-up view is shown of one portion of Fig. 3. Clearly, the chatter has been eliminated.

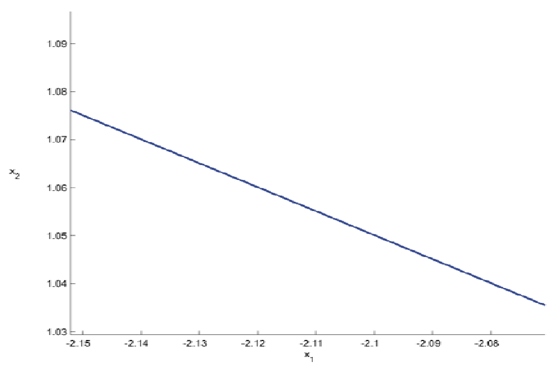

Figure 4: A trajectory under singular control.

Physical bounds on the control variable (bounded torque) influence the size of the controllable set. With a greater control bound, we may generally expect to be able to dictate that states far from the origin are ultimately driven asymptotically to the origin. With a smaller control bound (less torque) we generally observe that a smaller set of states surrounding the origin will ultimately be driven asymptotically to the origin. To illustrate this let $u_{max} = 1$ within the prior example, and note Fig. 5. The reduction in control bounds reduces the number of states that may be forced to the origin by the control algorithm. In fact, many states which approached the origin under $u_{max} = 10$ now approach one of two nonzero equilibrium points at approximately (-1.5,0) and (1.5,0)

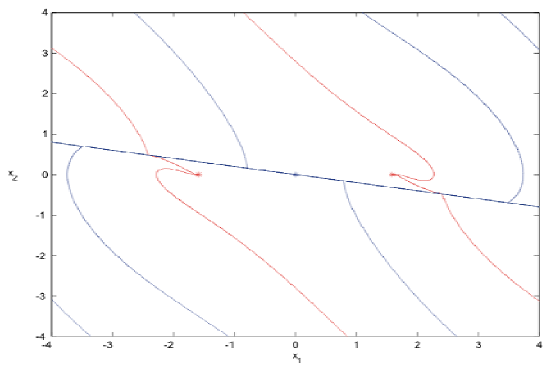

Figure 5: A trajectory under singular control (u_{max} = 1).

In addition, the placement of the switching surface, dictated by the values of k, b and m also impact algorithm performance. First, from (13) note that varying one of these parameters is equivalent to altering the cost function for the problem. A controller based upon the new parameters would then minimize the new cost in a quickest descent fashion. However, as long as the necessary conditions of [5] are met, the algorithm, based on the new parameters will still achieve the desired stability results. For example, consider letting u_{max} = 10 with b = 1 and m = 5. Figure 6 displays the new switching surface and the trajectories that asymptotically approach the origin.

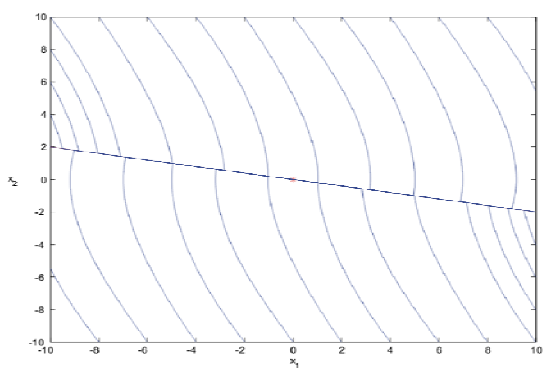

Figure 6: A trajectory under singular control (u_{max} = 10, b =1, m = 5).

Now let u_{max} = 10 with b = 3 and m = 1. Figure 7 displays the new switching surface and the trajectories that asymptotically approach the origin. For the choices of m and b considered thus far, the necessary conditions of [5] were satisfied and asymptotic convergence to the origin is achieved as trajectories reach the switching surface and ultimately slide along it until the origin.

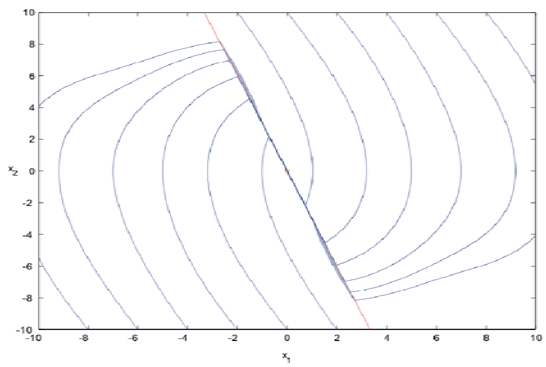

Figure 7: A trajectory under singular control (u_{max} = 10, b = 3, m = 1).

Suppose the necessary conditions were not satisfied (for example letting either b or m be less than zero). The dynamics of the switching surface would not be stable. This may be seen by setting b or m less than zero in (22). Referring to Theorem 2, if both m and b are less than zero the existence condition for reaching of the switching surface is not satisfied. That is,

$$\frac{\partial \sigma}{\partial x_2} u_{\max} = m u_{\max} < 0.$$

Therefore, if either or both m and b are less than zero, asymptotic stability cannot be achieved.

Conclusion

In this treatment, we have extended the applicability of Lyapunov optimizing control in terms of both stability and elimination of chatter. Using the QDC formulation, a control law was specified that explicitly considered cost and control bounds. Furthermore, a switching surface was specified and a singular control scheme was implemented. It is this singular control scheme that allows the analyst to exploit the stability properties of the singular surface but avoid the chattering phenomenon. Chatter has been shown in the literature to cause numerous undesired effects and should be avoided if possible. For the considered problem class, the derived methodology is an attractive scheme to resolve these concerns. The simulation results of the presented example illustrate the usefulness of the control law **QDCL**.

References

[1] Corless, M., and Leitmann, G., Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems, IEEE Transactions on Automatic Control, Vol. 26, No. 5, (1981), pp. 1139-1144.

[2] Grantham, W. J., Some Necessary Conditions for Steepest descent Controllability, Proceedings of the 1st American Controls Conference, Alexandria, VA, (1982), .

[3] Gutman, S., Uncertain Dynamical Systems – A Lyapunov Min-Max Approach, IEEE Transactions on Automatic Control, Vol. 24, No. 3, (1979), pp. 437-443.

[4] McDonald, D. B., Feedback Control Algorithms Through Lyapunov Optimizing Control and Trajectory Following Optimization, Ph.D. Dissertation, Mechanical and Materials Engineering, Washington State University, Pullman, WA, (2006).

[5] McDonald, D. B., Lyapunov Optimizing Sliding Mode Control for Linear Systems with Bounded Disturbance, Applied Mathematical Sciences, Vol. 2, No. 19, pp. 901-

918, Hikari Ltd., (2008).
[6] McDonald, D. B., Structurally Time Optimal Control for Linear Systems with Bounded Control and Constant Disturbance, Proceedings of the 26th American Controls Conference, New York, NY, (2007).
[7] Taha, E., Happawana, G., and Hurmuzlu, Y., Quantitative Feedback Theory (QFT) for Chattering Reduction and Improved Tracking in Sliding Mode Control (SMC), Journal of Dynamic Systems, Measurement, and Control, Vol. 125, No. 4, (2003), pp. 665-669.
[8] Vincent, T. L., and Grantham, W. J., Nonlinear and Optimal Control Systems, Wiley, N.Y., (1997).
[9] Vincent, T. L. and Grantham, W.J., Trajectory Following Methods in Control System Design, J. of Global Optimization, Vol. 23, (2002), pp. 267-282.
[10] Vincent, T. L., and Morgan, R. W., Guidance Against Maneuvering Targets Using Lyapunov Optimizing Feedback Control, in Proc. 21st Annual American Controls Conference, Anchorage, AK, (2002), pp. 215–220.
[11] Zlateva, P., Variable Structure Control Design for a Class of Nonlinear Uncertain Processes, Proceedings of the 1996 IEEE International Workshop on Variable Structure Systems, Tokyo, Japan, (1996), pp. 61-66.

Biography

DALE MCDONALD is an Assistant Professor within the McCoy School of Engineering at Midwestern State University in Wichita Falls, Texas. Dr. McDonald’s primary research interests are in the fields of optimal control and engineering education.