Paper #117, ENG 101

A Trajectory Following Control Algorithm for Chatter Elimination with Reduced Stiffness

Dale B. McDonald
McCoy School of Engineering
Midwestern State University
dale.mcdonald@mwsu.edu

Abstract

In this treatment a control algorithm is developed to zero the states of a class of nominally linear systems while considering accumulation of a state dependent cost. Due to physical limits on actuators and the presence of unknown disturbances that cannot be accounted for explicitly in the system mathematical model, the control problem addressed is actually nonlinear. Lyapunov optimizing control (LOC) techniques are used to specify a control law; LOC combines Lyapunov stability theory with function minimization to provide controls that consider stability of the origin and minimization of cost. LOC techniques, combined with trajectory following optimization (TFO) methods, provide a means to eliminate the chatter phenomenon which is notorious for the considered problem class. A control law that produces discontinuous chatter is undesirable; instantaneous switching of mechanical actuators can excite resonant modes, induce vibration, and produce a number of other detrimental effects. It has been shown that to provide a satisfactory system response using these methods a large gain must be introduced in an appropriately defined augmented system of governing differential equations. This creates a "stiff" set of differential equations which are difficult to treat analytically and numerically. To deal with this issue, an alternate TFO control implementation is proposed which greatly reduces the effect of the stiff differential equations. Ultimately, a control law that stabilizes the target, considers cost, and eliminates chatter in the presence of unknown disturbance is realized. In addition, the theory of LOC is extended by proving the stability results for instances when traditional LOC algorithms cannot.

Introduction

Control algorithms based upon Lyapunov's second method have proven quite effective for controlling linear and nonlinear systems subject to disturbances; excellent examples being [1] and [2]. Lyapunov optimizing control, which originated in [3], produces feedback controls by selecting a candidate Lyapunov function and choosing the control to minimize this function as much as possible along system trajectories [4], [5]. The advantage of LOC algorithms comes from a built in proof of their effectiveness. Assuming the target is the origin, if the candidate Lyapunov function is decreased everywhere outside the target, a sufficient condition for asymptotic stability is satisfied [6]. Furthermore, the LOC approach provides the analyst a means to design feedback controls where cost accumulation is explicitly considered and has been applied to many interesting problems [7, 8]. The difficulty in utilizing LOC methods arises during proof of asymptotic stability of the origin. For a dynamical system subject to a disturbance, without assumptions on the stability of the state equations, it may be quite difficult to guarantee that the candidate Lyapunov function decreases everywhere outside the origin. If the function doesn't decrease everywhere apart from the origin, stability cannot be proven.

Sliding mode control has proven to be an effective approach for the control of uncertain dynamical systems. The ability of these algorithms to reject disturbances and stabilize the origin has made SMC a subject of great interest in terms of both theoretical and practical research [9, 10]. The main idea of SMC is to select a switching surface in state space that is attractive, robust to disturbance, and consists of stable dynamics. Once trajectories reach this surface, they "slide" along it until the origin is reached. A primary drawback of SMC algorithms is their tendency to induce chatter; typified by high frequency (discontinuous) commutation of the control signal across this surface [11, 12]. This commutation may produce a variety of detrimental effects on a mechanical system such as excitement of resonant modes and extreme wear on actuators.

We propose a continuous control law based upon the techniques of trajectory following optimization. Rather than specify the control directly, a differential equation is defined for the time derivative of the control. This differential equation is integrated numerically, yielding a continuous control. The form of this differential equation is derived from an optimization perspective, synthesizing our desires of stability, cost considerations, and chatter elimination into a single, robust control law. The originality of this control is that this differential equation is derived from the optimal control necessary conditions applied to the candidate Lyapunov function, not by explicitly requiring the state to lie on the switching surface. Typically, such sliding mode controls are found by repeated time differentiation of the switching surface. In [13] a continuous control law was proposed to eliminate chatter; while successful, this required the introduction of a gain which created a stiff system of differential equations. We extend upon these results in two areas: 1) We prove finite time convergence to a properly defined switching surface and demonstrate stability through numerical simulation, and 2) We show that this modified TFO implementation greatly reduces the stiffness of the system of differential equations.

Problem Formulation

The cost functional is dependent upon the state **x**, and is given by

$$J = \int_0^{t_f} \Psi(x)dt \tag{1}$$

where

$$\Psi(x) = \frac{1}{2} x^T \frac{\partial^2 \Psi}{\partial x^2} x$$

The matrix $\partial^2\Psi/\partial x^T$ is constant, symmetric, and positive definite. We consider systems described by state equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}(u + v) \tag{2}$$

where the control variable *u* is an element of the constraint set

$$u_{\min} \le u \le u_{\max} \tag{3}$$

where $u_{\min} = -u_{\max}$ and the unknown disturbance *v* satisfies $|v| < u_{\max}$. We assume that the system (2) is in companion form; therefore

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

For convenience, define the $(n-1)\times 1$ vectors $\mathbf{x}_{\mathrm{p}}$ and $\mathbf{x}_{\mathrm{r}}$ such that

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_{\mathrm{r}} \\ \mathrm{x}_{\mathrm{n}} \end{bmatrix} = \begin{bmatrix} \mathrm{x}_1 \\ \mathbf{x}_{\mathrm{p}} \end{bmatrix} \tag{5}$$

With $\boldsymbol{a} = [a_{n1},\ldots, a_{nn}]$ and

$$\mathbf{A}_r = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6}$$

we may express (2) as

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{A}_r \mathbf{x}_p \\ \dot{x}_{\mathrm{n}} &= \mathbf{a}\mathbf{x} + \mathrm{u} + \mathrm{v} \end{aligned} \tag{7}$$

Our objective is to specify a control law $u(\mathbf{x})$ that drives the state to the origin and considers minimization of accumulated cost (1).

Quickest Descent Control and Trajectory Following Optimization

Quickest descent control [3, 4] is utilized to develop a new control law that considers cost while stabilizing the origin. In addition, combination of quickest descent control with trajectory following optimization (TFO) produces an algorithm that relieves the chatter phenomenon. Trajectory following methods solve optimization problems numerically by defining special sets of differential equations whose equilibrium solutions satisfy first-order necessary conditions. Using TFO to minimize cost will produce a control effort that avoids chatter while robust to disturbance.

Controls found by the application of QDC are derived by first selecting a descent function $\overline{\Psi}(\mathbf{x})$ that exhibits the following properties: 1) $\overline{\Psi}(\mathbf{0}) = 0$, 2) $\overline{\Psi}(\mathbf{x}) > 0 \;\forall\; \mathbf{x} \ne \mathbf{0}$, and 3) $\partial\overline{\Psi}/\partial\mathbf{x} \ne \mathbf{0} \;\forall\; \mathbf{x} \ne \mathbf{0}$. The control u for the problem (1)-(3) is chosen to decrease $\overline{\Psi}(\mathbf{x})$ as quickly as possible along trajectories $\mathbf{x}$(t). For that reason, u is chosen by

$$\min_u \frac{d\overline{\Psi}}{dt} \quad (8)$$

where we choose the cost accumulation rate $\Psi(\mathbf{x})$ as the descent function; that is we let $\Psi(\mathbf{x}) = \overline{\Psi}(\mathbf{x})$. Therefore, the control is chosen

$$\min_u \dot{\Psi} \quad (9)$$

where $\dot{\Psi} = d\Psi/dt$. If $\dot{\Psi} < 0 \forall \mathbf{x} \neq 0$ and properties (1) - (3) are satisfied, then we are assured that the origin is asymptotically stable.

Trajectory following methods solve optimization problems by defining special sets of differential equations whose equilibrium solutions satisfy first order necessary conditions. For this analysis, the minimization (9) is handled by TFO by using the gradient of the cost function, i.e. the time rate of change of *u* will be a function of $\partial\dot{\Psi}/\partial u$. For an excellent reference on control system design using TFO, see [6].

Necessary Conditions

In [13] necessary conditions are formulated on $\Psi(\mathbf{x})$ for stability of the target (origin). Necessary conditions are presented that guarantee trajectories will not converge to nonzero equilibrium points; conditions are given that also provide finite time convergence to the switching surface. In this treatment, it is assumed that these necessary conditions are satisfied; proof of stability for the new algorithm (reaching of the switching surface and demonstrated stability of the origin) despite the disturbance is the primary goal.

Properties of the Controller

The control is found through the minimization (9) where

$$\dot{\Psi} = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\left[\frac{\partial^2\Psi}{\partial\mathbf{x}^2}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\frac{\partial^2\Psi}{\partial\mathbf{x}^2}\right]\mathbf{x} + (\mathrm{u}+\mathrm{v})\frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{x}. \quad (10)$$

Application of (9) to (10) yields a preliminary control law (**PCL**)

$u(\mathbf{x}) =$	u_{max}	if	$\sigma(x) < 0$
	u_s	if	$\sigma(x) = 0$
	$u_{\min}$	if	$\sigma(x) > 0$

with

$$\sigma(\mathbf{x}) = \frac{\partial\dot{\Psi}}{\partial u} = \mathbf{b}^{\mathrm{T}}\frac{\partial^2\Psi}{\partial\mathbf{x}^2}\mathbf{x} \quad (11)$$

$$\frac{\partial\sigma}{\partial x} = \mathbf{b}^{\mathrm{T}}\frac{\partial^2\Psi}{\partial\mathbf{x}^2} \quad (12)$$

$$\dot{\sigma} = \mathbf{b}^{\mathrm{T}} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} \left[\mathbf{A}\mathbf{x} + \mathbf{b}(\mathrm{u} + \mathrm{v}) \right] \tag{13}$$

The control u_s is known as singular control [4]. In optimal control theory, additional necessary conditions are used for its derivation; the final singular control dictates that trajectories travel along the "switching surface" $S \equiv \{\mathrm{x} : \sigma(\mathrm{x}) = 0\}$. Trajectories generated by control laws of the form **PCL** are well known to chatter about a switching surface, due to the bang-bang (max-min) control effort.

Chattering Elimination Control with Reduced Stiffness

In this section a control law is proposed, denoted by **CECRS**, that stabilizes the origin and considers cost accumulation despite the presence of the unknown disturbance. The primary contribution of this treatment is to prove that **CECRS** accomplishes these objectives while eliminating chatter, and in a manner that is less "stiff" than those previously reported in the literature.

CECRS Formulation

Given the state equations (2) and the cost functional (1) we propose **CECRS**

$$\dot{u} = -\frac{1}{\varepsilon} sign(\sigma) \tag{14}$$

where

sign(σ) =	*-1*	if	σ(x) < 0
	0	if	σ(x) = 0
	1	if	σ(x) > 0

and ε is a small (singular perturbation) parameter. Note, that (14) is derived by defining a differential equation whose equilibrium solution satisfies first order necessary conditions for the optimization (9). Therefore, 1/ε is a system gain that results in an augmented set of state equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}(\mathrm{u} + \mathrm{v}) \\ \dot{u} &= -\frac{1}{\varepsilon} sign(\sigma) \end{aligned} \tag{15}$$

that exist on disparate time scales. We propose **CECRS** (14) rather than

$$\dot{u} = -\frac{1}{\varepsilon}(\sigma). \tag{16}$$

It is believed that **CECRS** will achieve the control objectives in a much less stiff manner. Specification of the time rate of change of the control law u, as in (14), eliminates discontinuous chatter (this will be shown through numerical simulation). The control **CECRS** has been specified as a function of the gradient of the cost function $(\partial \dot{\Psi} / \partial u) = \sigma$ as is typical with trajectory following methods.

Stability Results

First it will be shown that **CECSR** (14) produces trajectories that reach the switching surface S in finite time. Then we will demonstrate asymptotic stability through numerical simulation. In [9] it was shown for the system (2) with cost functional (1), initial trajectories satisfying

$$\frac{\partial \sigma}{\partial \mathbf{x}} u_{\max} > \left\| \frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{A} \right\| \left[C_1 \left\| x(t_i) \right\| + C_2 \left| \frac{\sigma[x(t_i)]}{\partial \sigma / \partial x_n} \right| \right] + \left\| \frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{b} v_{\max} \right\| \qquad (17)$$

where

$$C_1 = 1 + \left\| a_r \right\|$$

$$C_2 = 1 + C_1 \int_{t_i}^{\infty} \left\| e^{\Gamma(t-\tau)} \mathbf{b} \right\| d\tau$$

and

$$\mathbf{\Gamma} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{(\partial\sigma/\partial x_1)}{(\partial\sigma/\partial x_n)} & -\frac{(\partial\sigma/\partial x_2)}{(\partial\sigma/\partial x_n)} & \cdots & & \frac{(\partial\sigma/\partial x_{n-1})}{(\partial\sigma/\partial x_n)} \end{bmatrix}.$$

To prove finite time convergence to S, we need to show that **CECRS** (14) is able to produce the proper control u that saturates at its appropriate upper or lower bound, satisfying (17).

Theorem 1: Consider a trajectory that has overshot S by some small amount into the region $\sigma < 0$ with $u = u_{min}$ at time $t = t^*$. The time interval $(t - t^*)$ in which (14) produces the correct control saturation $u = u_{max}$ is a function of the parameter ε and the control bounds.

Proof: With

$$\dot{u} = -\frac{1}{\varepsilon} sign(\sigma)$$

we have

$$u(t) = u(t^*) - \frac{1}{\varepsilon} \int_{t^*}^{t} sign(\sigma) d\tau .$$

With $\sigma < 0$, $\text{sign}(\sigma) = -1$. Therefore

$$u(t) = u(t^*) + \frac{1}{\varepsilon} \int_{t^*}^{t} d\tau$$

which implies that

$$(t - t^*) = \varepsilon(u_{\max} - u_{\min}) \qquad (18)$$

and completes the proof.

The implication of Theorem 1 is that we may select the parameter ε such that the time interval in (18) provides an "initial" time $t = t_i$ that satisfies the inequality (17); trajectories generated by (14) reach S in finite time. Asymptotic stability of the origin, and the desired "less-stiff" behavior is shown through numerical simulation.

Numerical Simulation

The usefulness of the proposed control scheme **CECRS** is demonstrated by considering the problem of an inverted pendulum with a bounded disturbance. **CECRS** is compared to **PCL** and (16).

Example One

Consider the linearized inverted pendulum with disturbance input

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + v) \tag{19}$$

with the accumulated cost (1) given by $\Psi(\mathrm{x}) = 2.5\mathrm{x}_1^{\ 2} + \mathrm{x}_1\mathrm{x}_2 + \mathrm{x}_2^{\ 2}$; yielding $\sigma = x_1 + 2x_2$. In Fig. 1 and Fig. 2, the control law **PCL** was implemented and applied to the problem (19). In both the wide and close-up views, discontinuous chatter presents.

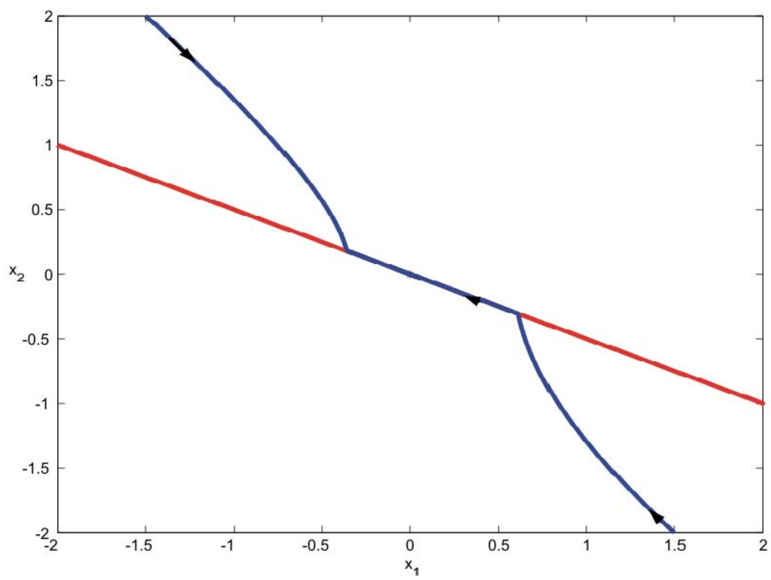

Figure 1: Typical trajectories under PCL.

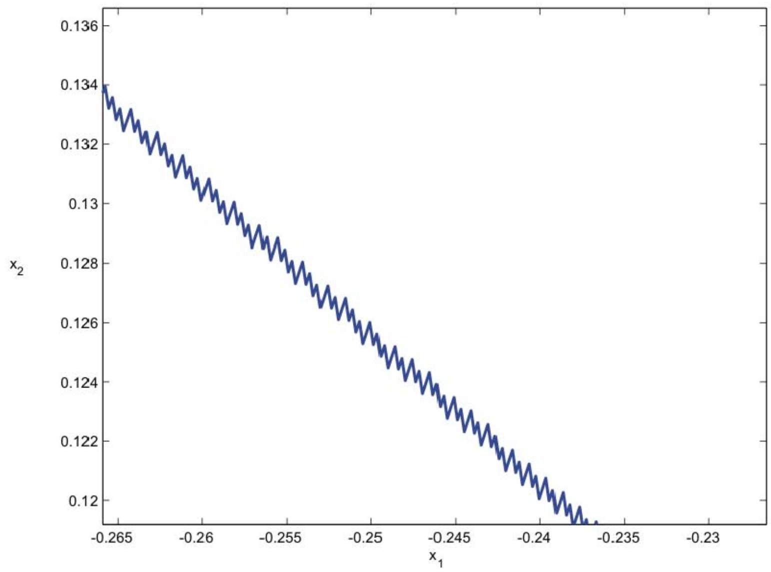

Figure 2: A chattering trajectory under PCL.

In Fig. 3, control laws (16) and **CECRS** were implemented and applied to the problem (19). Each eliminated chatter, but the time-scale characteristics differed greatly. Trajectories were generated using a 4-th order, fixed step-size, Runge-Kutta integration scheme with $\Delta t = 25$, 15,001 time steps, $u_{max} = 1$ and $v = .5\sin(t)$. For the control law (16), a parameter value $\varepsilon_1 = 0.00005$ was used. For **CECRS**, a parameter value $\varepsilon_2 = 0.01$ was used. The results are shown in Fig. 3 and Fig. 4. Trajectories generated by each algorithm look identical in the wide view of Fig. 3. However, in Fig. 4 the differences are illuminated. Much less oscillation about *S* is observed and the system of augmented differential equations is much less stiff as evidenced by $(\varepsilon_2/\ \varepsilon_1) = 200$.

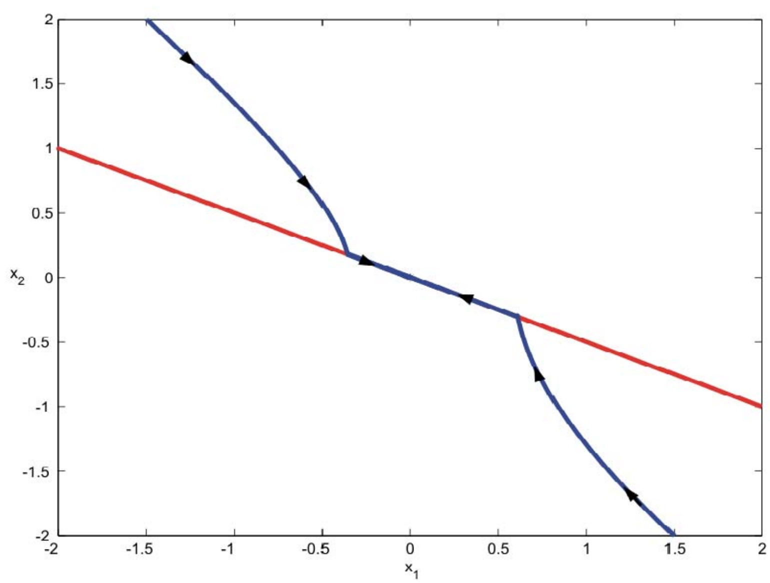

Figure 3: Typical trajectories for control laws (16) and **CECRS**.

ISBN 978-1-60643-379-9

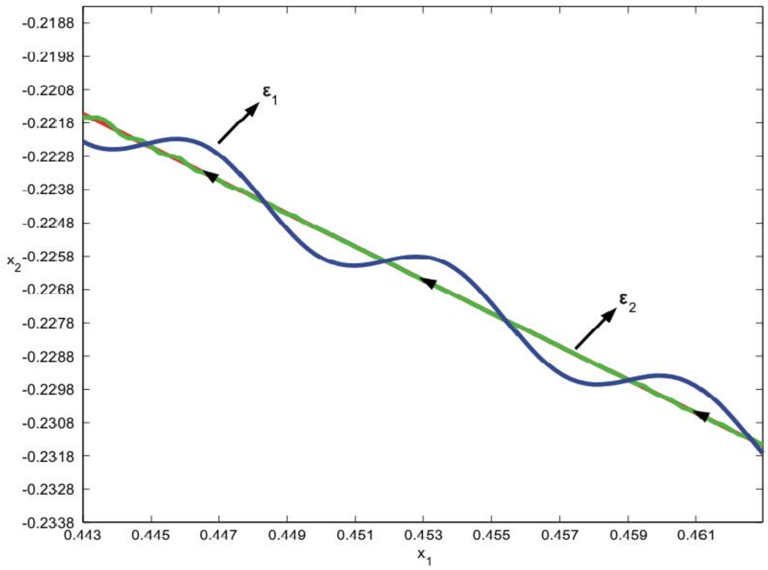

Figure 4: Typical trajectories for control laws (16) **CECRS**.

Conclusion

In this treatment, a control law was proposed that extends the applicability of Lyapunov optimizing control methods to systems subject to disturbances. Derivation of the control law using LOC methodology allowed explicit consideration of cost and stability of the origin, despite the presence of a bounded, but otherwise unknown disturbance. To counter the chattering phenomenon, TFO methods were applied to specify the final form **CECRS**. This new TFO formulation accomplished the desired results in a much less stiff manner than those previously reported in the literature.

References

[1] Corless, M., and Leitmann, G., Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems, IEEE Transactions on Automatic Control, Vol. 26, No. 5, (1981), pp. 1139-1144.

[2] Gutman, S., Uncertain Dynamical Systems – A Lyapunov Min-Max Approach, IEEE Transactions on Automatic Control, Vol. 24, No. 3, (1979), pp. 437-443.

[3] Grantham, W. J., Some Necessary Conditions for Steepest descent Controllability, Proceedings of the 1st American Controls Conference, Alexandria, VA, (1982).

[4] Vincent, T. L., and Grantham, W. J., Nonlinear and Optimal Control Systems, Wiley, N.Y., (1997).

[5] Vincent, T. L., and Morgan, R. W., Guidance Against Maneuvering Targets Using Lyapunov Optimizing Feedback Control, in Proc. 21st Annual American Controls Conference, Anchorage, AK, (2002), pp. 215–220.

[6] Vincent, T. L. and Grantham, W.J., Trajectory Following Methods in Control System Design, J. of Global Optimization, Vol. 23, (2002), pp. 267-282.
[7] McDonald, D. B., Feedback Control Algorithms Through Lyapunov Optimizing Control and Trajectory Following Optimization, Ph.D. Dissertation, Mechanical and Materials Engineering, Washington State University, Pullman, WA, (2006).
[8] McDonald D.B., and Grantham, W.J., Singular Perturbation Trajectory Following Algorithms for Min-Max Differential Games, in Advances in Dynamic Game Theory, Jørgensen, S. J., Quincampoix, M., and Vincent, T. L., eds., Birkhaüser, Boston, (2007), pp. 659 – 678.
[9] Utkin, V., I., Variable Structure Systems with Sliding Modes, IEEE Transactions on Automatic Control, Vol. 22, No. 2, (1977), pp. 212-222.
[10] Xiong, Y., Liu, J., Shin, K., and Zhao, W., On the Modeling and Optimization of Discontinuous Network Congestion Control Systems, Proceedings of the 23rd Annual Joint Conference of the IEEE Computer and Communications Societies, Hong Kong, (2004).
[11] Taha, E., Happawana, G., and Hurmuzlu, Y., Quantitative Feedback Theory (QFT) for Chattering Reduction and Improved Tracking in Sliding Mode Control (SMC), Journal of Dynamic Systems, Measurement, and Control, Vol. 125, No. 4, (2003), pp. 665-669.
[12] Zlateva, P., Variable Structure Control Design for a Class of Nonlinear Uncertain Processes, Proceedings of the 1996 IEEE International Workshop on Variable Structure Systems, Tokyo, Japan, (1996), pp. 61-66.
[13] McDonald, D. B., Lyapunov Optimizing Sliding Mode Control for Linear Systems with Bounded Disturbance, Applied Mathematical Sciences, Vol. 2, No. 19, pp. 901-918, Hikari Ltd., (2008).

Biography

DALE MCDONALD is an Assistant Professor within the McCoy School of Engineering at Midwestern State University in Wichita Falls, Texas. Dr. McDonald’s primary research interests are in the fields of optimal control and engineering education.