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Stability of Quasiperiodic Systems

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Abstract

In this paper, we present some observations on the stability of a special class of quasiperiodic systems. In quasiperiodic system, the periodicity of parametric excitation is incommensurate with the periodicity of the certain terms multiplying state vector. We present a Lyapunov type approach and the Lyapunov- Floquet (LF) transformation to derive stability conditions. This approach can be utilized to investigate the robustness, stability margin and design controller for the system.

Introduction

A large class of engineering systems, such as, structures subjected to quasiperiodic are described by linear ordinary differential equations with time varying coefficients. These linear systems, in general, are described as

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}; \tag{1}$$

where $\mathbf{A}(t)$ is an $n \times n$ quasiperiodic matrix and $\mathbf{y}$ is an n dimensional vector. In general, it is not a trivial problem to determine if equation (1) is asymptotically stable, simply stable or unstable. The researchers have used perturbation type techniques to analyze this equation or numerical approaches to investigate the stability of this system [1-3].

In this work, we address the stability of a special class of quasiperiodic systems called as periodic quasiperiodic systems where equation (1) can be written as

$$\dot{\mathbf{y}} = [\mathbf{A}_0(t) + \varepsilon \mathbf{A}_1(t)]\mathbf{y}; \tag{2}$$

Where $\mathbf{A}_0(t)$ has the principal period T and $\varepsilon \mathbf{A}_1(t)$ has the period T_1. It is noted that these periods are incommensurate. These types of equations arise in parametrically excited Micro Electro Mechanical Systems (MEMS) [4]. It is noted that ε is a small parameter indicating that the magnitude of the quasiperiodic component in small compared to the periodic component. It is also noted that $\mathbf{A}(t)$ has a strong parametric excitation. In this paper we present the methodology to investigate the stability of the system given by equation (2) using the Lyapunov-Floquet (L F) Transformation.

This paper is organized as follows. In section 2, a brief mathematical background on the L-F transformation is provided. Section 3 discusses the stability conditions followed by an example. The discussion and conclusions are presented in section 4.

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2. Mathematical Background

2.1 Floquet Theory and L-F Transformation

Consider equation (1), if $\varepsilon = 0$ i.e. the system is purely time periodic then the state transition matrix (STM) $\mathbf{\Phi}(t)$ of equation (1) can be factored as [5]

$$\mathbf{\Phi}(t) = \mathbf{Q}(t)e^{\mathbf{R}t}, \mathbf{Q}(t) = \mathbf{Q}(t+2T), \mathbf{Q}(0) = \mathbf{I} \tag{3}$$

where the matrix $\mathbf{Q}(t)$ is real and periodic with period $2T$, $\mathbf{R}$ is an $n \times n$ real time invariant matrix and $\mathbf{I}$ is the identity matrix. Matrix $\mathbf{Q}(t)$ is known as the Lyapunov–Floquet (L-F) transformation matrix [5].

The transformation $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ produces a real; time invariant representation of purely time periodic system (equation (1) with $\varepsilon = 0$) given by

$$\dot{\mathbf{z}}(t) = \bar{\mathbf{A}}\,\mathbf{z}(t) \tag{4}$$

It is to be noted that matrix $\bar{\mathbf{A}}$ in equation (4) is time invariant.

2.2 Construction of Lyapunov Functions

Lyapunov's direct method is widely used in the stability analysis of general dynamical systems. It makes use of a Lyapunov function $V(x,t)$. This scalar function of the state and time may be considered as some form of time dependent generalized energy. The basic idea of the method is to utilize the time rate of energy change in $V(x,t)$ for a given system to judge whether the system is stable or not. The details about Lypunov's method and stability theorems can be found in reference [6].

For a linear system with constant coefficients are concerned, it is rather simple to find a Lyapunov function. Consider the linear system

$$\dot{\underset{\sim}{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\underset{\sim}{\mathbf{x}}(t) \tag{5}$$

where $\mathbf{A}$ is a constant matrix. A quadratic form of $V(\mathbf{x})$ may be assumed as

$$V(\underset{\sim}{\mathbf{x}}) = \underset{\sim}{\mathbf{x}}^T \mathbf{P} \underset{\sim}{\mathbf{x}}$$

where $\mathbf{P}$ is a real, symmetric and positive definite matrix. Then

$$\dot{V}(\underset{\sim}{\mathbf{x}}) = \dot{\underset{\sim}{\mathbf{x}}}^T \mathbf{P}\underset{\sim}{\mathbf{x}} + \underset{\sim}{\mathbf{x}}^T \mathbf{P}\dot{\underset{\sim}{\mathbf{x}}} = (\tilde{\mathbf{A}}\underset{\sim}{\mathbf{x}})^T \mathbf{P}\underset{\sim}{\mathbf{x}} + \underset{\sim}{\mathbf{x}}^T \mathbf{P}\tilde{\mathbf{A}}\underset{\sim}{\mathbf{x}} \tag{6}$$

or

$$\dot{V}(\underset{\sim}{\mathbf{x}}) = \underset{\sim}{\mathbf{x}}^T (\tilde{\mathbf{A}}^T \mathbf{P} + \mathbf{P}\tilde{\mathbf{A}})\underset{\sim}{\mathbf{x}} \tag{7}$$

According to the Lyapunov theorem for autonomous systems, if $\dot{V}(x)$ is negative definite then the null solution is asymptotically stable [6]. Therefore, one can write

$$\tilde{\mathbf{A}}^T \mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} = -\mathbf{C} \tag{8}$$

where $\mathbf{C}$ is a positive definite matrix. Equation (8) is called the Lyapunov equation. It has been shown by Kalman and Bertram [7] that if has eigenvalues with negative real parts (asymptotically stable), then for every given positive definite matrix $\mathbf{C}$, there exists a unique Lyapunov matrix $\mathbf{P}$. In this study, matrix $\mathbf{C}$ is always taken as the identity matrix.

3. Stability of periodic quasiperiodic systems

Consider the periodic quasiperiodic linear differential equation given by equation (2). In order to determine the stability bounds on $\mathbf{A}_1(t)$, we first assume that time-periodic part of equation (2) [$\mathbf{A}_0(t)$] is asymptotically stable. Using the L-F transformation $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$, equation (2) can be written as

$$\dot{\mathbf{z}} = [\bar{\mathbf{A}} + \varepsilon \mathbf{G}(t)]\mathbf{z} \tag{9}$$

where $\mathbf{G}(t) = \mathbf{Q}^{-1}(t)\mathbf{A}_1(t)\mathbf{Q}(t)$. It is to be noted that $\bar{\mathbf{A}}$ is a constant matrix whose eigenvalues have negative real parts. We follow the approach presented by Infante [8] to obtain the stability bounds.

Theorem-1 [8]: If, for some positive definite matrix $\mathbf{B}$ and some $\bar{\varepsilon} > 0$,

$$E\{\lambda_{\max}[\bar{\mathbf{A}}^T + \mathbf{G}(t)^T + \mathbf{B}[\bar{\mathbf{A}} + \mathbf{G}(t)]\mathbf{B}^{-1}] \le -\bar{\varepsilon} \tag{10}$$

then equation (9) is almost surely asymptotically stable in the large.

Proof: Consider the quadratic (Lyapunov) function $V(\mathbf{z}) = \mathbf{z}^T\mathbf{B}\mathbf{z}$. Then along the trajectories of equation (9), define

$$\lambda(t) = \frac{\dot{V}(\mathbf{z})}{V(\mathbf{z})} = \frac{\mathbf{z}^T[(\bar{\mathbf{A}} + \mathbf{G}(t))^T\mathbf{B} + \mathbf{B}(\bar{\mathbf{A}} + \mathbf{G}(t))]\mathbf{z}}{\mathbf{z}^T\mathbf{B}\,\mathbf{z}} \tag{11}$$

From the properties of pencils of quadratic forms [9] we can obtain the inequality

$$\lambda_{\min}[(\bar{\mathbf{A}} + \mathbf{G}(t))^T + \mathbf{B}(\bar{\mathbf{A}} + \mathbf{G}(t))\mathbf{B}^{-1}] \le \lambda(t) \le \lambda_{\max}[(\bar{\mathbf{A}} + \mathbf{G}(t))^T + \mathbf{B}(\bar{\mathbf{A}} + \mathbf{G}(t))\mathbf{B}^{-1}] \tag{12}$$

where $\lambda_{\max}$ and $\lambda_{\min}$, are the maximum and minimum real eigenvalues of a pencil. It follows from equation (11) and equation (12) that

$$V[\mathbf{z}(t)] = V[\mathbf{z}(t_0)]e^{\int_{t_0}^{t}\lambda(\tau)d\tau} \equiv V[\mathbf{z}(t_0)]e^{(t-t_0)[\frac{1}{t-t_0}\int_{t_0}^{t}\lambda(\tau)d\tau]} \tag{13}$$

It can be observed that, if $E\{\lambda(t)\} \le -\bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$, $V[\mathbf{z}(t)]$ is bounded and that $V[\mathbf{z}(t)] \to 0$ as $t \to \infty$. This is the condition imposed by inequality given by (10), which proves the results. Since $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ the stability of equation (9) implies the stability of equation (2).

It is remarked that a necessary condition for inequality (10) to hold is that the eigenvalues of matrix $\bar{\mathbf{A}}$ have negative real parts. It is also possible to obtain a result which is easier to compute but not as sharp.

Corollary: If, for some positive definite matrix B and some $\bar{\varepsilon} > 0$,

$$E\{\lambda_{\max}[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}]\} \le -\lambda_{\max}[\bar{\mathbf{A}}^T + \mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}] - \bar{\varepsilon} \tag{14}$$

then equation (9) is almost surely asymptotically stable in the large.

Proof: The proof follows immediately from theorem by noting that

$$\lambda(t) \le \lambda_{\max}[(\bar{\mathbf{A}} + \mathbf{G}(t))^T + \mathbf{B}(\bar{\mathbf{A}} + \mathbf{G}(t))\mathbf{B}^{-1}] \le \lambda_{\max}[\bar{\mathbf{A}}^T + \mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}] + \lambda_{\max}[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}], \tag{15}$$

The second inequality being obtained by performing two maximization separately. Further, using $E\{\ \}$ operator

$$E\{\lambda(t)\} \le \lambda_{\max}[\bar{\mathbf{A}}^T + \mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}] + E\{\lambda_{\max}[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}]\} \le \bar{\varepsilon} \qquad (16)$$

yields the desired result. It is obvious that, unless the second inequality in (15) is an equality, the stability results obtained will not as good as those given by the theorem. It is noted that this theorem and corollary can be extended to study stability and robustness of a linear time-periodic system subjected to random perturbations in a straightforward fashion and for the details, we refer the reader to reference 10.

Example

Consider the system

$$\dot{\mathbf{y}} = [\tilde{\mathbf{A}}(t) + \varepsilon\hat{\mathbf{A}}(t)]\mathbf{y} \qquad (17)$$

where

$$\tilde{\mathbf{A}}(t) = \omega\begin{bmatrix} -1+\alpha\cos^2(\omega t) & 1-\alpha\sin(\omega t)\cos(\omega t) \\ -1-\alpha\sin(\omega t)\cos(\omega t) & -1+\alpha\sin^2(\omega t) \end{bmatrix}, \hat{\mathbf{A}}(t) = \begin{bmatrix} 0 & 0 \\ -f(t) & 0 \end{bmatrix}$$

α is a system parameter and $\omega = 2\pi$. The state transition matrix (STM), $\mathbf{\Phi}(t)$ of this system [11] is given as

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{(\alpha-1)\omega t}\cos(\omega t) & e^{-\omega t}\sin(\omega t) \\ -e^{(\alpha-1)\omega t}\sin(\omega t) & e^{-\omega t}\cos(\omega t) \end{bmatrix} = \mathbf{Q}(t)e^{\mathbf{R}t} \qquad (18)$$

Factoring the state transition matrix as shown above, the Lyapunov-Floquet transformation matrix $\mathbf{Q}(t)$ is found as

$$\mathbf{Q}(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}, \quad e^{\mathbf{R}t} = \begin{bmatrix} e^{(\alpha-1)\omega t} & 0 \\ 0 & e^{-\omega t} \end{bmatrix} \qquad (19)$$

It is noted that the system is unstable for all $\alpha > 1$. Using the L-F transformation $\mathbf{z}(t) = \mathbf{Q}(t)\mathbf{y}(t)$ (c. f. equation (19)) equation (17) to yields a time-invariant system given by

$$\dot{\mathbf{z}}(t) = \bar{\mathbf{A}}\mathbf{z}(t) \qquad (20)$$

Let $\mathbf{V} = \mathbf{z}^T(t)\mathbf{B}\mathbf{z}(t)$, where $\mathbf{B}$ is a constant, symmetric, positive definite matrix.

Then
$$\dot{\mathbf{V}} = \dot{\mathbf{z}}^T\mathbf{B}\mathbf{z} + \mathbf{z}^T\mathbf{B}\dot{\mathbf{z}} = \mathbf{z}^T[\bar{\mathbf{A}}^T\mathbf{B} + \mathbf{B}\bar{\mathbf{A}}]\mathbf{z} \equiv -\mathbf{z}^T\mathbf{C}\mathbf{z} \qquad (21)$$

Setting
$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix}, \mathbf{C} = \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (22)$$

substituting equations (22) into equation (21), yields $B_{11} = -\frac{1}{2\omega(\alpha-1)}$ $(\alpha < 1)$, $B_{12} = 0$ and $B_{22} = \frac{1}{2\omega}$.

Therefore,

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{2\omega(\alpha-1)} & 0 \\ 0 & \frac{1}{2\omega} \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} -2\omega(\alpha-1) & 0 \\ 0 & 2\omega \end{bmatrix} \qquad (23)$$

Since $B_{11} > 0$ for $\alpha < 1$ and

$$Det(\mathbf{B}) = \begin{vmatrix} -\frac{1}{2\omega(\alpha-1)} & 0 \\ 0 & \frac{1}{2\omega} \end{vmatrix} = -\frac{1}{2\omega^2(\alpha-1)} > 0 \qquad (24)$$

Therefore, $\mathbf{B}$ is a positive definite symmetric matrix and Lyapunov stability conditions are satisfied.

Once $\mathbf{B}$ matrix is constructed, the stability theorem and the corollary can be used to determine the stability conditions for the system. Simple computations yield

$$\mathbf{D} = \bar{\mathbf{A}}^T + \mathbf{G}^T(t) + \mathbf{B}[\bar{\mathbf{A}} + \mathbf{G}(t)]\mathbf{B}^{-1}$$

$$= \begin{bmatrix} 2\omega(\alpha-1) + 2f(t)\sin(\omega t)\cos(\omega t) & \dfrac{-[(\alpha-2)\cos^2(\omega t)+1]f(t)}{\alpha-1} \\ [(\alpha-2)\cos^2(\omega t)+1]f(t) & -2\omega - 2f(t)\sin(\omega t)\cos(\omega t) \end{bmatrix} \tag{25}$$

Setting $Det[\mathbf{D}-\lambda\mathbf{I}]=0$, the eigenvalues λ of the $\mathbf{D}$ matrix are computed as

$$\lambda_{1,2} = -\omega(2-\alpha) \pm \sqrt{\omega^2\alpha^2 + \frac{1}{\alpha-1}[2\omega\alpha(\alpha-1)f(t)\sin(\omega t) + (2\alpha\cos^2(\omega t) - \alpha^2\cos^4(\omega t) - 1)f^2(t)]} \tag{26}$$

Application of the theorem yields

$$E\{\lambda_{\max}[\mathbf{D}]\} = -\omega(2-\alpha)$$

$$+E\{\sqrt{\omega^2\alpha^2 + \frac{1}{\alpha-1}[2\omega\alpha(\alpha-1)f(t)\sin(\omega t) + (2\alpha\cos^2(\omega t) - \alpha^2\cos^4(\omega t) - 1)f^2(t)]}\} \le 0 \tag{27}$$

or,

$$E\{\sqrt{\omega^2\alpha^2 + \frac{1}{\alpha-1}[2\omega\alpha(\alpha-1)f(t)\sin(\omega t) + (2\alpha\cos^2(\omega t) - \alpha^2\cos^4(\omega t) - 1)f^2(t)]}\} \le \omega(2-\alpha) \tag{28}$$

Using Schwarz's Inequality [9], $(E\{f(t)\})^2 \le E\{f^2(t)\}$, one obtains

$$2\omega\alpha(\alpha-1)E\{f(t)\}E\{\sin(2\omega t) + (2\alpha E\{\cos^2(\omega t)\} - \alpha^2 E\{\cos^4(\omega t) - 1)E\{f^2(t)\} \le 4\omega^2(1-\alpha)^2 \tag{29}$$

Since $E\{\sin(2\omega t)\}=0$, $E\{\cos^2(\omega t)\} = \dfrac{1}{2}$ and $E\{\cos^4(\omega t)\} = \dfrac{3}{8}$, inequality (29)yields

$$E\{f^2(t)\} \le \frac{32\omega^2(1-\alpha)^2}{8+3\alpha^2-8\alpha} \tag{30}$$

The results obtained from condition (30) for α from 0 to 1 are shown in Fig. 1.

In order to get the conditions for almost sure asymptotic stability from the corollary, matrices $[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}]$ and $[\bar{\mathbf{A}} + \mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}]$ are calculated as

$$\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1} = f(t)\begin{bmatrix} 2\sin(\omega t)\cos(\omega t) & -\dfrac{(\alpha-2)\cos^2(\omega t)+1}{\alpha-1} \\ (\alpha-2)\cos^2(\omega t)+1 & -2\sin(\omega t)\cos(\omega t) \end{bmatrix}$$

$$\bar{\mathbf{A}} + \mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1} = \begin{bmatrix} 2(\alpha-1)\omega & 0 \\ 0 & -2\omega \end{bmatrix} \tag{31}$$

The maximum eigenvalues of matrices given by (31) are computed as

$$\lambda_{\max}[\mathbf{G}^T(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}] = |f(t)|\sqrt{\frac{1-2\alpha\cos^2(\omega t)+\alpha^2\cos^4(\omega t)}{1-\alpha}} \tag{32}$$

$$\lambda_{\max}[\bar{\mathbf{A}}^{\mathbf{T}}+\mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}] = -2(1-\alpha)\omega$$

Applying the corollary

$$E\{\lambda_{\max}[\mathbf{G}^T(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}] \le -\lambda_{\max}[\bar{\mathbf{A}}^T+\mathbf{B}\bar{\mathbf{A}}\mathbf{B}^{-1}]-\varepsilon \tag{33}$$

yields

$$E\{|f(t)|\sqrt{\frac{1-2\alpha\cos^2(\omega t)+\alpha^2\cos^4(\omega t)}{1-\alpha}}\} \le 2(1-\alpha)\omega \tag{34}$$

Then using Schwarz's Inequality in equation (34), one obtains

$$E\left\{f^2(t)\left[\frac{1-2\alpha\cos^2(\omega t)+\alpha^2\cos^4(\omega t)}{1-\alpha}\right]\right\} \le 4(1-\alpha)^2\omega^2 \tag{35}$$

or

$$E\{f^2(t)\}[1-2\alpha E\{\cos^2\omega t\}+\alpha^2 E\{\cos^4\omega t\}] \le 4(1-\alpha)^3\omega^2 \tag{36}$$

Since $E\{\cos^2(\omega t)\}=\frac{1}{2}$ and $E\{\cos^4(\omega t)\}=\frac{3}{8}$, inequality (36) provides the condition for almost sure asymptotic stability from corollary as

$$E\{f^2(t)\} \le \frac{32(1-\alpha)^3\omega^2}{8-8\alpha+3\alpha^2} \tag{37}$$

As expected, condition (37) is weaker than condition(30). Fig. 1 displays the result obtained from equation (37) for α in the range of 0 to 1. A comparison of conditions yielding from the theorem and the corollary is also shown in Fig. 1.

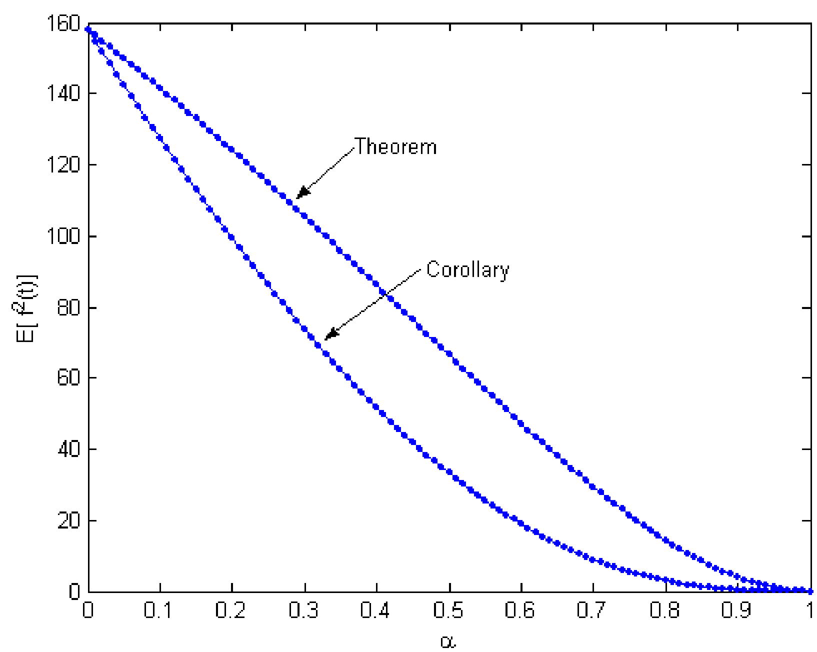

Figure 1: Stability results for example 1 obtained by the Theorem and Corollary

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4. Discussions and Conclusions

In this paper, simple and efficient computational techniques to guarantee sufficient conditions for almost sure asymptotic stability of periodic quasiperiodic systems have been presented. First, the Lyapunov-Floquet transformation has been utilized to convert the periodic part of time-periodic system to a time-invariant form. For the linear periodic-quasiperiodic system, a theorem and related corollary have been suggested using the results previously obtained by Infante [8]. In order to apply the theorem and the corollary successfully, it is observed that the eigenvalues of matrix $\bar{\mathbf{A}}$, which governs the stability of the system, must have negative real parts and matrix $\mathbf{B}$ must be positive definite. One example is presented to show the application. It is expected that this methodology would be useful in studying stability and designing controllers for a number of MEMS where governing differential equations have time periodic quasiperodic coefficients. The approaches presented in this paper can be extended to study stability and robustness of a linear time-periodic system subjected to random perturbations.

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Biography

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